## Combinatorics

Counting

An Overview

- Introductory Example
- What to Count
  - Lists
  - Permutations
  - Combinations.
- The Basic Principle
- Counting Formulas
- The Binomial Theorem.
- Partitions
- Solutions

## Example

As I was going to St. Ives

I met a man with seven wives

Every wife had seven sacks

Every sack had seven cats

Every cat had seven kits

Kits, cats, sacks, wives

How many were going to St. Ives?

*Ans*: None.

How many were going the other ways?

7 Wives.

$7 \times 7 = 49$ sacks.

$49 \times 7 = 343$ cats.

$343 \times 7 = 2401$ kits.

Total $= 2800$.

## Lists

Can be Counted Easily

*Order Pairs*: $(x, y) = (w, z)$ iff $w = x$ and $z = y$.

*Ordered Triples*: $(x, y, z) = (u, v, w)$ iff $u = x,\ v = y$, and $w = z$.

*Lists of Length $r$ (AKA Order $r$-tuples)*:

$$(x_1, \cdots, x_r) = (y_1, \cdots, y_s)$$

iff $s = r$ and $y_i = x_i$ for $i = 1, \cdots, r$.

*Example: License Plates.* A license plate has the form $LMNxyz$, where

$$L, M, N \in \{A, B, \cdots, Z\},$$
$$x, y, z \in \{0, 1, \cdots, 9\}$$

and, so, is a list of length six.

## Basic Principle of Combinatorics
## The Multiplication Principle

**For Two**: If there are $m$ choices for $x$ and then $n$ choices for $y$, then there are $m \times n$ choices for $(x, y)$.

**For Several**: If there are $n_i$ choices for $x_i$, $i = 1, \cdots, r$, then there are

$$n_1 \times n_2 \times \cdots \times n_r$$

choices for $(x_1, \cdots, x_r)$.

*Example.* There are $7^3 = 7 \times 7 \times 7 = 343$ choices for $(\text{wife}, \text{sack}, \text{cat})$.

*Example.* There are

$$26^3 \times 10^3 = 17,576,000$$

license plates. Of these

$$26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11,232,000$$

have distinct letters and digits (no repetition).

## Permutations

A *permutation* of length $r$ is a list $(x_1, \cdots, x_r)$ with distinct components (no repetition); that is, $x_i \neq x_j$ when $i \neq j$.

*Examples.* $(1, 2, 3)$ is a permutation of three elements; $(1, 2, 1)$ is a list, but not a permutation

**Counting Formulas**. From $n$ objects,

$$n^r = n \times \cdots \times n \ (r \text{ factors})$$

lists of length $r$ and

$$(n)_r := n \times (n-1) \times \cdots \times (n-r+1)$$

permutations of length $r$ may be formed.

*Examples*: There are $10^3 = 1000$ three digit numbers of which $(10)_3 = 10 \times 9 \times 8 = 720$ list distinct digits.

**Some Notation**. Recall

$$(n)_r = n \times (n-1) \times \cdots \times (n-r+1)$$

positive integers $n$ and $r$.

**Factorials**: When $r = n$, write

$$n! = (n)_n = n \times (n-1) \times \cdots \times 2 \times 1.$$

*Conventions*: $(n)_0 = 1$ and $0! = 1$.

*Notes a).* The book only considers $r = n$.

*b).* $(n)_r = 0$ if $r > n$.

*c).* If $r < n$, then

$$n! = (n)_r(n-r)!$$

## Examples

*Example.* A group of 9 people may choose officers (P,VP,S,T) in $(9)_4 = 3024$ ways.

*Example.* 7 books may be arranged in $7! = 5040$ ways.

If there are 4 math books and 3 science books, then there are

$$2 \times 4! \times 3! = 288$$

arrangements in which the math books are together and the science books are together.

## Combinations

A combination of size $r$ is a set $\{x_1, \cdots, x_r\}$ of $r$ distinct elements. Two combinations are equal if they have the same elements, possibly written in different orders.

*Example.* $\{1,2,3\} = \{3,2,1\}$, but $(1,2,3) \neq (3,2,1)$.

*Example.* How many committees of size 4 may be chosen from 9 people? Choose officers in two steps:

Choose a committee in ?? ways.

Choose officers from the committee in 4! ways.

From the Basic Principle

$$(9)_4 = 4! \times ??.$$

So,

$$?? = \frac{(9)_4}{4!} = 126.$$

## Combinations Formula

From $n \geq 1$ objects,

$$\binom{n}{r} = \frac{1}{r!}(n)_r$$

combinations of size $r \leq n$ may be formed.

*Example.*

$$\binom{9}{4} = \frac{1}{4!}(9)_4 = 126.$$

*Proof*: Replace 9 and 4 by $n$ and $r$ in the example.

*Example: Bridge.* A bridge hand is a combination of $n = 13$ cards drawn from a standard deck of $N = 52$. There are

$$\binom{52}{13} = 635,013,559,600$$

such hands.

## Binomial Coefficients

*Alternatively*:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

**The Binomial Theorem**: For all $-\infty < x, y < \infty$,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

*Example.* When $n = 3$,

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

*Proof.* If

$$(x+y)^n = (x+y) \times \cdots \times (x+y)$$

is expanded, then $x^r y^{n-r}$ will appear as often as $x$ can be chosen from $r$ of the $n$ factors; i.e., in

$$\binom{n}{r}$$

ways.

## Binomial Identities

**Recall**:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

*Examples -a).* Setting $x = y = 1$,

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

*b).* Letting $x = -1$ and $y = 1$,

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r = 0$$

for $n \geq 1$.

## Partitions

**AKA Divisions**

An Example

Q: How many distinct arrangements can be formed from the letters

MISSISSIPPI?

A: There are 11 letters which may be arranged in

$$11! = 39,916,800 \qquad (*)$$

ways, but this leads to double counting. If the 4 "S"s are permuted, then nothing is changed. Similarly, for the 4 "I"s and 2 "P"s. So, (*) the each configuration of letters

$$4! \times 4! \times 2! = 1,152$$

times and the answer is

$$\frac{11!}{4! \times 4! \times 2!} = 34,650.$$

## Partitions

Defintions

Let $Z$ be a set with $n$ elements. If $r \geq 2$ is an integer,then an *ordered partition of $Z$ into $r$ subsets* is a list

$$(Z_1, \cdots, Z_r)$$

where $Z_1, \cdots, Z_r$ are mutually exclusive subsets of $Z$ whose union is $Z$; that is,

$$Z_i \cap Z_j = \emptyset \text{ if } i \neq j$$

and

$$Z_1 \cup \cdots \cup Z_r = Z.$$

Let

$$n_i = \#Z_i,$$

the number of elements in $Z_i$. Then

$$n_1, \cdots, n_r \geq 0$$

and

$$n_1 + \cdots + n_r = n.$$

*Example.* In the "MISSISSIPPI" Example, 11 positions,

$$Z = \{1, 2, \cdots, 11\}$$

were partitioned into four groups of sizes

$$n_1 = 4 \text{ “I”s}$$

$$n_2 = 1 \text{ “M”s}$$

$$n_3 = 2 \text{ “P”s}$$

$$n_4 = 4 \text{ “S”s}$$

*Example.* In a bridge game, a deck of 52 cards is partitioned into four hands of size 13 each, one for each of South, West, North, and East.

## The Partitions Formula

Let $n$, $r$, and $n_1, \cdots, n_r$ be integers for which

$$n, r \geq 1,$$

$$n_1, \cdots, n_r \geq 0,$$

$$n_1 + \cdots + n_r = n.$$

If $Z$ is a set of $n$ elements, then there are

$$\binom{n}{n_1, \cdots, n_r} := \frac{n!}{n_1! \times \cdots \times n_r!}$$

ways to partition $Z$ into $r$ subsets $(Z_1, \cdots, Z_r)$ for which $\#Z_i = n_i$ for $i = 1, \cdots, r$.

*Example.*

$$\binom{11}{4, 1, 2, 4} = 34,650.$$

**Def.** Called *multinomial coefficients*

## The Number of Solutions

If $n$ and $r$ are positive integers, how many integer solutions to the equations

$$n_1, \cdots, n_r \geq 0$$

$$n_1 + \cdots + n_r = n$$

are there?

*First Warm Up Example.* How many arrangements from $a$ A's and $b$ B's–for example, ABAAB)? There are

$$\binom{a+b}{a} = \binom{a+b}{b}$$

such, since an arrangement is determined by the $a$ places occupied by A.

## The Number of Solutions

Continued

*Second Warm Up Example.* Suppose $n = 8$ and $r = 4$. Represent solutions by $o$ and " + " by $|$. For example,

$$ooo|oo||ooo$$

means

$$n_1 = 3,$$

$$n_2 = 2,$$

$$n_3 = 0,$$

$$n_4 = 3.$$

*Note*: Only $r - 1 = 3$ $|$'s are needed.

There are as many solutions as there are ways to arrange $o$ and $|$. By the last example, there are

$$\binom{8+3}{3} = \binom{11}{3} = 165$$

solutions.

## A General Formula

If $n$ and $r$ are positive integers, then there are

$$\binom{n+r-1}{r-1} = \binom{n+r-1}{n}$$

integer solutios to

$$n_1, \cdots, n_r \geq 0$$

$$n_1 + \cdots + n_r = n.$$

If $n \geq r$, then there are

$$\binom{n-1}{r-1}$$

solutions with

$$n_i \geq 1$$

for $i = 1, \cdots, r$.

## Combinatorics

Summary

- Lists, permuatations, and combinations.
- The Basic Principle
- Counting Formulas

  Lists $n^r$

  Permuations $(n)_r$

  Combinations $\binom{n}{r}$

  Partitions $\binom{n}{n_1, \cdots, n_r}$

  Solutions $\binom{n+r-1}{r-1}$