11 (a) from $\{1, 2, \cdots, (10)^3\}$, the largest number can be divisible by 3 is 999, by 5 is 10^3, by 7 is 994, by 15 is 990, by 105 is 945, the smallest number can be divisible by 3 is 3, by 5 is 5 by 7 is 7, by 15 is 15, by 105 is 105, the the number of interger N will be divisible by 3 is $\frac{999+3}{3} - 1 = 333$, by 5 is $\frac{10^3+5}{5} - 1 = 200$, by 7 is $\frac{994+7}{7} - 1 = 142$, by 15 is $\frac{945+15}{15} - 1 = 66$, by 105 is $\frac{945+105}{105} - 1 = 9$. $\therefore$ $p(\text{divisible by } 3) = \frac{333}{1000}$, $p(\text{divisible by } 7) = \frac{142}{1000}$, $p(\text{divisible by } 15) = \frac{66}{1000}$

$p(\text{divisible by } 5) = \frac{200}{1000}$, $p(\text{divisible by } 105) = \frac{9}{1000}$

if 10^3 is replaced by 10^k, then the largest number can be divisible by 3 is between $10^k - 3 \sim 10^k$ by 5 is between $10^k - 5 \sim 10^k$, by 7 is between $10^k - 7 \sim 10^k$, by 15 is between $10^k - 15 \sim 10^k$ by 105 is between $10^k - 105 \sim 10^k$. then the number of interger N will be divisible by 3 is between $\frac{10^k - 3 + 3}{3} - 1 = \frac{10^k - 3}{3}$ and $\frac{10^k}{3}$, by 5 is between $\frac{10^k - 5}{5}$ and $\frac{10^k}{5}$, by 7 is between $\frac{10^k - 7}{7}$ and $\frac{10^k}{7}$, by 15 is between $\frac{10^k - 15}{15}$ and $\frac{10^k}{15}$, by 105 is between $\frac{10^k - 105}{105}$ and $\frac{10^k}{105}$

so the limiting distribution $p(\text{divisible by } 3) \geq \frac{10^k - 3}{3\, 10^k} \to \frac{1}{3}$ and $p(\text{visible by } 3) \leq \frac{10^k}{3 \cdot 10^k} = \frac{1}{3}$

$\therefore$ $p(\text{divisible by } 3) = \frac{1}{3}$, by the same method, we get $p(\text{divisible by } 5) = \frac{1}{5}$, $p(\text{divisible by } 7) = \frac{1}{7}$, $p(\text{divisible by } 15) = \frac{1}{15}$. $p(\text{divisible by } 105) = \frac{1}{105}$

13. X could have the possible value of 0, 500, 1000, 1500, 2000

$p(X=0) = p(\text{no sale on first and no sale on second}) = (.7)(.4) = .28$

$p(X=500) = p(1 \text{ sale and it is for standard}) = \frac{1}{2} \cdot p(1 \text{ sale}) = \frac{1}{2}\left[p(\{\text{sale, no sale}\}) + p(\{\text{no sale, sale}\})\right] = \frac{1}{2}\left[(.3)(.4) + (.7)(.6)\right] = .27$

$p(X=1000) = p(2 \text{ standard sales}) + p(1 \text{ sale for deluxe}) = \frac{1}{4} p(2 \text{ sales}) + \frac{1}{2} p(1 \text{ sale})$.

$= \frac{1}{4}(.6)(.3) + .27 = .315$

$p(X=1500) = p(2 \text{ sales, One deluxe and one standard}) = \frac{1}{2}(.3)(.6) = .09$

$p(X=2000) = p(2 \text{ sales, both deluxe}) = (.3)(.6)\,1/4 = .045$

20. X denote the gambler's winnings.

$X = 1$.	first bet win and (lose, win, win).	$p = \frac{18}{38} + \frac{20}{38}\left(\frac{18}{38}\right)^2$
$X = -1$	(lose, win, lose), (lose, lose, win)	$p = \frac{20}{38}\left(\frac{18}{38}\right)\left(\frac{20}{38}\right) \times 2$
$X = -3$	(lose, lose, lose)	$p = \left(\frac{20}{38}\right)^3$

(a). $p(X>0) = p(X=1) = 18/38 + (20/38)(18/38)^2 = .5918$

(b). No, because if the gambler wins then he or she wins \$1, However, a lose would either be \$1 or \$3

(c) $E(X) = 1[18/38 + (20/38)(18/38)^2] - 1[(20/38)2(20/38)(18/38)] - 3[(20/38)^3] = -.108$

22. (a) Let N denote the number of games played, for $i=2$.

$N=2$. if team A win all the two games or team B win all the two games. probability $= p^2 + (1-p)^2$

$N=3$. if team A win the last game and team B win the first or the second game. or the same for team B. probability $= 2p(1-p)\cdot p + 2(1-p)p(1-p) = 2(1-p)p$

$\therefore E(N) = 2[p^2 + (1-p)^2] + 3[2p(1-p)] = 2 + 2p(1-p)$.

To get the maximum value, take the derivative of $E(N)$ $\dfrac{dE(N)}{dp} = 2-4p = 0 \therefore p = \frac{1}{2}$.

(b) for $i=3$

$N=3$ if team A win all the 3 games or the same for team B probability $= p^3 + (1-p)^3$

$N=4$ if team A win the last game and team B win one of the first 3 games or the same for team B probability $= 3p^3(1-p) + 3p(1-p)^3$

$N=5$. if team A win the last game and team B win two of the first 4 games or the same for team B. probability $= 6p^3(1-p)^2 + 6p^2(1-p)^3$

$$E(N) = 3[p^3 + (1-p)^3] + 4[3p^3(1-p) + 3p(1-p)^3] + 5[6p^3(1-p)^2 + 6p^2(1-p)^3]$$

$\dfrac{dE(N)}{dp} = 24p^3 - 36p^2 + 6p + 3$. its value at $p = \frac{1}{2}$ is easily seen to be 0.

30 First $E(X) = \sum_{n=1}^{\infty} 2^n \cdot (1/2)^n = \sum_{n=1}^{\infty} 1 = \infty$

(a). I would probably not to pay \$1 million to play this game once. because if we want to win more than \$1 million, then $2^n \geq 10^6$ $n \geq \dfrac{6 \log 10}{\log 2}$ the probability to win is $\frac{1}{2^n} \leq \frac{1}{10^6}$ is too small.

(b) If you could play an arbitrary large number of games., you keep on playing until you win. a large number of dollars one time (2^n, $n \to \infty$) then, it is surely that you will win.

32. If T is the number of tests needed for a group of 10 people, then

$T=1$ if the test is negative, the probability is $(.9)^{10}$

$T=11$ if the test is positive, the probability is $(1-(.9)^{10})$

$\therefore E(T) = (.9)^{10} + 11[1-(.9)^{10}] = 11 - 10(.9)^{10}$

54 X denote the number of cars abandoned weekly, X from possion distribution with parameter 2.2.

$\therefore p(X=x) = \dfrac{e^{-2.2}(2.2)^x}{x!}$

(a) p (no abandoned cars) $= p(X=0) = e^{-2.2}$

(b) p (at least two abandoned cars) $= p(X \geq 2) = 1 - p(X=0) - p(X=1) = 1 - 3.2e^{-2.2}$

55 X_1 denote the number of errors when typed by the first typist, X_2 denote the number of error by the second typist, $X_1 \sim$ poisson(3), $X_2 \sim$ poisson(4.2)

p(no errors) $= p$(no errors | First typist) p(First typist) $+ p$(no errors | second typist) p(second typist)

$= p(X_1=0)\cdot\frac{1}{2} + p(X_2=0)\cdot\frac{1}{2} = \frac{1}{2}(e^{-3} + e^{-4.2})$

76 Let E denote the event that the mathematician first discovers that the right-hand matchbox is empty and there are k matches in the left-hand box at the time. Now, the event will occur if and only if the (N_2+1)th choice of the right-hand matchbox is made at the (N_1+N_2+1-k) trial. Hence it's from negative binomial distribution. with $p=\frac{1}{2}$ $r=N_2+1$, $n=N_1+N_2+1-k$

$\therefore p(E) = \binom{N_1+N_2-k}{N_2}\left(\frac{1}{2}\right)^{N_1+N_2-k+1}$. the same for the event that first discovers the left-hand matchbox is empty $p(E') = \binom{N_1+N_2-k}{N_1}\left(\frac{1}{2}\right)^{N_1+N_2-k+1}$.

$\therefore p = \binom{N_1+N_2-k}{N_2}\left(\frac{1}{2}\right)^{N_1+N_2-k+1} + \binom{N_1+N_2-k}{N_1}\left(\frac{1}{2}\right)^{N_1+N_2-k+1}$

77 Let E denote the event that ~~the mathematician~~ at the moment the right-hand matchbox is empty, there are k matches in the left-hand box. Now the event will occur if and only if the N th choice of the right-hand matchbox is made at $(2N-k)$ trial. it is also from negative binomial distribution with $p=\frac{1}{2}$ $r=N$. $n=2N-k$ $\therefore p(E) = \binom{2N-k-1}{N-1}\left(\frac{1}{2}\right)^{2N-k}$

$\therefore p = 2\binom{2N-k-1}{N-1}\left(\frac{1}{2}\right)^{2N-k}$

6 the right-hand side $\sum_{i=1}^{\infty} P\{N \geq i\} = \sum_{i=1}^{\infty}\sum_{k=i}^{\infty} P\{N=k\} = \sum_{k=1}^{\infty}\sum_{i=1}^{k} P\{N=k\} = \sum_{k=1}^{\infty} k\,P\{N=k\} = E(N)$

7. the left-hand side $\sum_{i=0}^{\infty} i\,p\{N>i\} = \sum_{i=1}^{\infty} i \sum_{k=i+1}^{\infty} p\{N=k\} = \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \cdot i\,p\{N=k\}$

$$= \sum_{k=1}^{\infty} p\{N=k\} \sum_{i=0}^{k-1} i = \sum_{k=1}^{\infty} p\{N=k\} \cdot \frac{k(k-1)}{2} = \left[\sum_{k=1}^{\infty} k^2 p\{N=k\} - \sum_{k=1}^{\infty} k p\{N=k\}\right]\frac{1}{2}$$

$$= \frac{1}{2}\left(E[N^2] - E[N]\right)$$

15. First prove $\sum_{i=0}^{[n/2]} \binom{n}{2i} p^{2i} q^{n-2i} = \frac{1}{2}\left[(p+q)^n + (q-p)^n\right]$

the right-hand side $\frac{1}{2}\left[(p+q)^n + (q-p)^n\right] = \frac{1}{2}\left[\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k} + \sum_{k=0}^{n}\binom{n}{k}(-p)^k q^{n-k}\right]$

$$= \frac{1}{2}\left[\sum_{k=0}^{[n/2]}\binom{n}{2k}p^{2k}q^{n-2k} + \cancel{\sum_{k=0}^{[n/2]}\binom{n}{2k+1}p^{2k+1}q^{n-2k-1}} + \sum_{k=0}^{[n/2]}\binom{n}{2k}(-p)^{2k}q^{n-2k} + \cancel{\sum_{k=0}^{[n/2]}\binom{n}{2k+1}(-p)^{2k+1}q^{n-2k-1}}\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{[n/2]}\binom{n}{2k}p^{2k}q^{n-2k} + \sum_{k=0}^{[n/2]}\binom{n}{2k}(-p)^{2k}q^{n-2k}\right] = \sum_{k=0}^{[n/2]}\binom{n}{2k}p^{2k}q^{n-2k}$$

Then Let X denote the number of heads when ~~toss~~ the coin n times. , then $X \sim \text{Binomial}(n,p)$

$$p(X = \text{even number}) = \sum_{k=0}^{[n/2]}\binom{n}{2k}p^{2k}(1-p)^{n-2k} = \frac{1}{2}\left[(p+1-p)^n + (1-p-p)^n\right] = \frac{1}{2}\left[1+(q-p)^n\right]$$

20. Let S denote the number of heads that occur when all n coins are tossed, and note that S has a distribution that is approximately that of a possion random variable with mean λ. Then, because X is distributed as the conditional distribution of S given that $S>0$

$$p(X=1) = p(S=1 \mid S>0) = \frac{p(S=1)}{p(S>0)} \approx \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}$$

25. the number of events that occur in a specific time is a Poisson random variable with parameter λ and the number of events is counted is a Binomial random variable $B(n,p)$.

so $p(m \text{ counted}) = \sum_{n=m}^{\infty} p(m \mid n \text{ events})\, p(n \text{ events})$

$$= \sum_{n=m}^{\infty}\binom{n}{m}p^m(1-p)^{n-m} e^{-\lambda}\cdot \lambda^n/n!$$

$$= e^{-\lambda p}\frac{(\lambda p)^m}{m!}\sum_{n=m}^{\infty}\frac{[\lambda(1-p)]^{n-m}}{(n-m)!}e^{-\lambda(1-p)}$$

$$= e^{-\lambda p}\frac{(\lambda p)^m}{m!}\underbrace{\sum_{n=0}^{\infty}\frac{[\lambda(1-p)]^n}{n!}e^{-\lambda(1-p)}}_{=1}$$

$$= e^{-\lambda p}\frac{(\lambda p)^m}{m!}$$

So the number of events that are counted is a Poisson random variable with parameter λp

Intuitively, the Poisson λ random variable arises as the approximate number of successes in n independent trials each having a small success probability $\frac{\lambda}{n}$. Now if each successful trial is counted with probability p, the number counted is Binomial with parameter $\frac{\lambda}{n}p$

which is approximately Poisson with parameter $\frac{\lambda}{n}p \cdot n = \lambda p$.

if $\lambda=10$, $p=\frac{1}{50}$, $X \sim \text{Poisson}(\lambda p) = \text{Poisson}(\frac{1}{5})$

(a) $p(X=1) = (e^{-\frac{1}{5}})\frac{1}{5}$ (b) $p(X \geq 1) = 1 - p(X=0) = 1 - e^{-\frac{1}{5}}$

(c) $p(X \leq 1) = p(X=0) + p(X=1) = e^{-\frac{1}{5}} + e^{-\frac{1}{5}} \cdot \frac{1}{5} = \frac{6}{5} e^{-\frac{1}{5}}$

26. if $n=0$, the right-hand side $\frac{1}{0!}\int_\lambda^\infty e^{-x}dx = (-e^{-x})\Big|_\lambda^\infty = e^{-\lambda}$ = left-hand side.

Assume. $n=k$., the equation $\sum_{i=0}^{k} e^{-\lambda}\frac{\lambda^i}{i!} = \frac{1}{k!}\int_\lambda^\infty e^{-x}x^k dx$

Then for $n=k+1$. $\frac{1}{(k+1)!}\int_\lambda^\infty e^{-x} \cdot x^{k+1} dx = \frac{1}{(k+1)!}\left[[-e^{-x} \cdot x^{k+1}]\Big|_\lambda^\infty + \int_\lambda^\infty e^{-x}(k+1)x^k dx\right.$

$$= \frac{1}{(k+1)!}\left[e^{-\lambda} \cdot \lambda^{k+1} + \frac{(k+1)!}{k!}\int_\lambda^\infty e^{-x}x^k dx\right]$$

$$= \frac{1}{(k+1)!}\left[e^{-\lambda} \cdot \lambda^{k+1} + (k+1)!\sum_{i=0}^{k} e^{-\lambda}\frac{\lambda^i}{i!}\right]$$

$$= \sum_{i=0}^{k+1} e^{-\lambda}\frac{\lambda^i}{i!}$$

so we proved the equation now $\sum_{i=0}^{n} e^{-\lambda}\frac{\lambda^i}{i!} = \frac{1}{n!}\int_\lambda^\infty e^{-x}x^n dx$

27. $\because$ X is a geometric random variable, $p(X=n) = p(1-p)^{n-1}$.

$$\therefore p\{X=n+k \mid X>n\} = \frac{p\{X=n+k\}}{p\{X>n\}} = \frac{p(1-p)^{n+k-1}}{\sum_{k=n}^{\infty} p(1-p)^{k}} = \frac{p(1-p)^{n+k-1}}{p\,\frac{(1-p)^{n-1}}{1-(1-p)}} = p(1-p)^{k-1}$$

$= p\{X=k\}$.

If the first n trials are fall failures, then it is as if we are beginning anew at that time

28. The events $\{X>n\}$ and $\{Y<r\}$ are both equivalent to the event that there are fewer than r successes in the first n trials, hence, they are the same event.

31. Let Y denote the largest of the remaining m chips., the event is the same if you draw m chips from $n+m$ chips and those m chips have the largest value. $Y=j$ so the other $m-1$ chips are drawn from $j-1$ numbers $\therefore p(Y=j) = \binom{j-1}{m-1}\Big/\binom{n+m}{m}$, $m \leq j \leq n+m$.

Let X denote the number of chips drawn. having numbers that exceed all the numbers of those remaining, so $X = n+m-Y$.

$p(X=i) = p(Y=n+m-i) = \binom{m+n-i-1}{m-1}\Big/\binom{n+m}{m}$, $i \leq n$.

34. Let X denote the number of elements in the chosen subset, then $P(X=k) = \dfrac{\binom{n}{k}}{2^n - 1}$

$\therefore E(X) = \sum_{k=0}^{n} \dfrac{k\binom{n}{k}}{2^n-1}$ using the result of theoretical exercise 12 of Chapter 1

$$= \frac{n2^{n-1}}{2^n-1}$$

$$E(X^2) = \sum_{k=0}^{n} \frac{k^2\binom{n}{k}}{2^n-1} = \frac{2^{n-2}n(n+1)}{2^n-1}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{(2^n-1)2^{n-2}n(n+1) - n^2 2^{2n-2}}{(2^n-1)^2} = \frac{n2^{2n-2} - n(n+1)2^{n-2}}{(2^n-1)^2} \sim \frac{n2^{2n-2}}{2^{2n}}$$

$$= \frac{n}{4}$$

Additional problem:

$$p(X_1=\ell) = \sum_{n=\ell}^{\infty} p(X_1=\ell \mid X=n)\,p(X=n) = \sum_{n=\ell}^{\infty} \binom{n}{\ell} p^{\ell}(1-p)^{n-\ell} \cdot \frac{e^{-\lambda}\lambda^n}{n!} = \frac{e^{-\lambda p}(\lambda p)^{\ell}}{\ell!}$$

$$p(X_2=m) = \sum_{n=m}^{\infty} p(X_2=m \mid X=n)\,p(X=n) = \sum_{n=m}^{\infty} \binom{n}{m}(1-p)^m p^{n-m} \frac{e^{-\lambda}\lambda^n}{n!} = \frac{e^{-\lambda(1-p)}[\lambda(1-p)]^m}{m!}$$

$$p(X_1=\ell, X_2=m) = p(X_1=\ell, X=m+\ell) = p(X_1=\ell \mid X=m+\ell)\,p(X=m+\ell)$$

$$= \binom{m+\ell}{\ell} p^{\ell}(1-p)^m \cdot \frac{e^{-\lambda}\lambda^{m+\ell}}{(m+\ell)!} = \frac{e^{-\lambda p}(\lambda p)^{\ell}[\lambda(1-p)]^m \cdot e^{-\lambda(1-p)}}{\ell!\,m!} = p(X_1=\ell)\,p(X_2=m)$$

$\therefore X_1$ and X_2 are independent random variables.