P_{249} - P_{250}

16. Let X denote the annual rainfall in a certain region and $X \sim N(40, 4^2)$

the probability that a year has not a rainfall of over 50 inches in $p(X \leq 50) =$

$1 - p\left(\frac{X-40}{4} > \frac{50-40}{4}\right) = \Phi(2.5) = .9938$.

Hence, $(p(X < 50))^{10} = (.9938)^{10}$ is the final answer.

The assumption is the annual rainfalls are independent random variables.

20. Let X denote the number of people in favor. Then X is binomial with $n=100$, $p=.65$.

Since $np(1-p) = 100(.65)(.35) > 10$. Then X is approximately normal with mean 65

standard deviation $\sqrt{(.65)(.35)\times 100} \approx 4.77$. Also Let Z be a standard normal random variable

(a) $p(X \geq 50) = p(X \geq 49.5) = p((X-65)/4.77 \geq (49.5-65)/4.77) \approx p(Z \geq -3.25) \approx .9994$

(b) $p(59.5 \leq X \leq 70.5) \approx p((59.5-65)/4.77 \leq X \leq (70.5-65)/4.77) = 2p(Z \leq 1.15) - 1 \approx .75$

(c) $p(X \leq 74.5) \approx p(Z \leq 9.5/4.77) \approx .977$

21 Let X denote the height, in inches, of a 25-year-old man. Then $X \sim N(71, 6.25)$

then $p(X \geq 6$ feet 2 inches$) = p(X \geq 74) = p\left(\frac{X-71}{2.5} \geq \frac{74-71}{2.5}\right) = 1 - \Phi(1.2) = 1 - .8849$

$= .1151$

$p(X \geq 6$ foot 5 inches $\mid X \geq 6$ foot$) = \dfrac{p(X \geq 77)}{p(X \geq 72)} = \dfrac{1-\Phi(2.4)}{1-\Phi(0.4)} =$

$$= \frac{1 - .9918}{1 - .6554} = 0.0245$$

26 Let X denote the number of heads in the 1000 tosses.

If the coin is fair, then $X \sim$ Binomial $(1000, .5)$ and using the normal approximation of X

$\sim N(500, 250)$ (since $\mu = 1000 \times (.5) = 500$ and $\sigma^2 = 1000 \times (.5) \times (.5) = 250$), the probability of

making false conclusion is $p(X \geq 525) = 1 - p(X \leq 524) \approx 1 - \Phi\left(\frac{524-500}{\sqrt{250}}\right) = 1 - \Phi(1.5179)$

$= 1 - .9357 = 0.0643$

If the coin is biased, then $X \sim$ Bin $(1000, .55)$ and the normal approximation is $X \sim$

$N(550, 1000(.55)(.45))$. Thus the probability of making a wrong conclusion is now

$p(X \leq 524) \approx \Phi\left(\frac{524-550}{\sqrt{1000(.55)(.45)}}\right) = \Phi(-1.65) = 1 - \Phi(1.65) = 1 - .9505 = 0.0495$

27. Let X denote the times the coin landed heads. if the coin is fair, $X \sim$ Binomial$(1000, .5)$

and approximately normal $(500, 250)$

then $p(X \geq 5800) = p(X > 5799.5) = p\left(Z > \frac{799.5}{\sqrt{250}}\right) = p(Z > 15.99) \approx 0$

so it is reasonable to say the coin is unfair.

2. (a) V is symmetric about 0 if and only if the distribution of V is the same as $-V$

then $\forall t$ $P(V \le t) = \int_{-\infty}^{t} f(v)\,dv = P(-V \le t) = \int_{-t}^{\infty} f(v)\,dv = \int_{-\infty}^{t} f(-v)\,dv$.

take the derivatives $f(t) = f(-t)$ $\forall t$

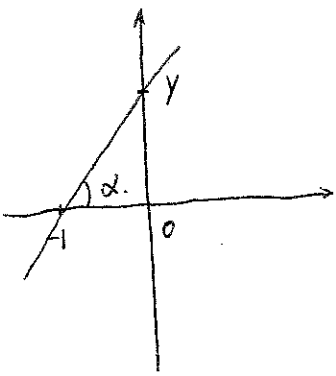
(b) $EV = \int_{-\infty}^{\infty} v \cdot f(v)\,dv = \int_{0}^{\infty} v f(v)\,dv + \int_{-\infty}^{0} v f(v)\,dv = \int_{0}^{\infty} v f(v)\,dv + \int_{0}^{\infty} -v f(-v)\,dv = 0$

$$E(V^{2m+1}) = \int_{0}^{\infty} v^{2m+1} f(v)\,dv + \int_{-\infty}^{0} v^{2m+1} f(v)\,dv$$

$$= \int_{0}^{\infty} v^{2m+1} f(v)\,dv + \int_{\infty}^{0} -v^{2m+1} f(-v)\,d(-v)$$

$$= \int_{0}^{\infty} v^{2m+1} [f(v) - f(-v)]\,dv = 0$$ for any integer m.

3



Let $\tan\alpha$ denote the slope of a line through point $(-1, 0)$ and through point $(0, y)$, then $y = \tan\alpha$ and $\alpha \sim \text{Uniform}(0, \pi)$

then $P(Y \le y) = P(\tan\alpha \le y) = P(\alpha \le \arctan y) = \frac{1}{\pi}\arctan y$.

$$f(y) = \frac{d}{dy}\left(\frac{1}{\pi}\arctan y\right) = \frac{1}{\pi(1+y^2)}$$

so y has the standard Cauchy density.

4. if y follows the Cauchy density $C(\lambda, \mu)$ $f(y) = \frac{1}{\pi}\frac{\lambda}{\lambda^2 + (y-\mu)^2}$. $-\infty < y < \infty$.

then then density function of $\frac{1}{y} = x$ is $f(x) = \frac{1}{\pi}\frac{\lambda}{\lambda^2 + (\frac{1}{x}-\mu)^2} \cdot \frac{1}{x^2}$

$$f(x) = \frac{1}{\pi}\frac{\lambda}{\lambda^2 x^2 + (\mu x - 1)^2} = \frac{1}{\pi}\cdot\frac{\lambda}{(\lambda^2+\mu^2)x^2 - 2\mu x + 1} = \frac{\lambda}{\pi(\lambda^2+\mu^2)}\,\frac{1}{x^2 - \frac{2\mu}{\lambda^2+\mu^2}x + \frac{\mu^2}{(\lambda^2+\mu^2)^2} - \frac{\mu^2}{(\lambda^2+\mu^2)^2} + \frac{1}{\lambda^2+\mu^2}}$$

$$= \frac{\lambda}{\pi(\lambda^2+\mu^2)}\,\frac{1}{(x - \frac{\mu}{\lambda^2+\mu^2})^2 + \cancel{\frac{1}{\lambda^2+\mu^2}}\,\frac{\mu^2}{(\lambda^2+\mu^2)^2}}$$

$$= \frac{\lambda}{\pi(\lambda^2+\mu^2)}\,\frac{1}{\frac{\lambda^2}{(\lambda^2+\mu^2)^2} + (x - \frac{\mu}{\lambda^2+\mu^2})^2}$$

so $\frac{1}{y}$ follows the Cauchy density $C\left(\frac{\lambda}{\lambda^2+\mu^2}, \frac{\mu}{\lambda^2+\mu^2}\right)$

5 (a) If X is a random variable with density f and CDF F, then $\forall y \in (0,1)$, we can find x. $y = F(x)$.

$p(F(X) \le y) = p(F(X) \le F(x)) = p(X \le x) = F(x) = y$.

so $F(x) \sim \text{Uniform}(0,1)$

(b). Let $\tilde{X} = F^{-1}(u)$, u is uniform $(0,1)$, then $\forall y \in \mathbb{R}$.

$p(\tilde{X} \le y) = p(F^{-1}(u) \le y) = p(u \le F(y)) = F(y)$.

$\therefore$ density $f(\tilde{x}) = \frac{d}{dy} p(\tilde{x} \le y) = f(y)$. is the same as the density function of x.

6 (a). First prove $E(X^{2k}) = (2k-1)E(X^{2k-2})$. $X \sim N(0,1)$.

$$E(X^{2k-2}) = \int_{-\infty}^{\infty} x^{2k-2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \left[\frac{x^{2k-1}}{2k-1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right]\Bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{x^{2k-1}}{2k-1} (-x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{x^{2k}}{2k-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{2k-1} E(X^{2k})$$

$\because E(X^2) = 1$. $\therefore$ $E(X^{2k}) = (2k-1)E(X^{2k-2}) = (2k-1)(2k-3)E(X^{2k-4})$

$\ldots\ldots = (2k-1)!!\, E(X^2) = (2k-1)!!$ $\forall k \in N$.

$$E(X^{2k+1}) = \int_{-\infty}^{\infty} x^{2k+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_0^{\infty} x^{2k+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + \int_{-\infty}^{0} x^{2k+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_0^{\infty} x^{2k+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + \int_{\infty}^{0} -x^{2k+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} d(-x)$$

$$= \int_0^{\infty} x^{2k+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - \int_0^{\infty} x^{2k+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0$$

(b) Let X denote the random variable produced after the coin toss.

$$X = \begin{cases} T & \text{coin lands Head.} \\ -T & \text{coin lands tail} \end{cases}$$

if $x > 0$ $p(X \le x) = p(X \le x | H) p(H) + p(X \le x | T) p(T)$

$$= \frac{1}{2} \cdot \int_0^x \lambda e^{-\lambda x} dx + \frac{1}{2} = \frac{1}{2} + \frac{1}{2}(1 - e^{-\lambda x}) = 1 - \frac{1}{2} e^{-\lambda x}$$

if $x < 0$ $p(X \le x) = p(X \le x | H) p(H) + p(X \le x | T) p(T)$

$$= p(X \le x | T) p(T) = \frac{1}{2} \cdot \int_{-\infty}^{x} \lambda e^{-\lambda x} dx = \frac{1}{2} e^{\lambda x}$$

the density function $f(x) = \begin{cases} \frac{1}{2} \lambda e^{-\lambda x} & x > 0 \\ \frac{1}{2} \lambda e^{\lambda x} & x < 0 \end{cases}$