15. (a) $\iint_{(x,y)\in R} f(x,y)\,dx\,dy = 1 = \iint_{(x,y)\in R} c\,dx\,dy = c\,A(R)$ where $A(R)$ is the area of the region R.

then $A(R) = \frac{1}{c}$

(b) if (X, Y) is uniformly distributed over the square centered at $(0,0)$, whose sides are of length 2, then $A(R) = 2\times 2 = 4$ $c = \frac{1}{4} = f(x,y)$ if $-1 \le x, y \le 1$.

then $f(x) = \int_{-1}^{1} f(x,y)\,dy = \int_{-1}^{1} \frac{1}{4}\,dy = \frac{1}{2}$, $f(y) = \int_{-1}^{1} f(x,y)\,dx = \int_{-1}^{1} \frac{1}{4}\,dx = \frac{1}{2}$

then $f(x,y) = f(x) f(y)$ which means X and Y are independent

and each is distributed uniformly over $(-1,1)$

(c) $P(X^2+Y^2 \le 1) = \frac{1}{4}\iint_{\{x^2+y^2\le 1\}} dx\,dy = \frac{1}{4}$ (area of circle) $= \pi/4$

17. the probability that X_2 lies between X_1 and X_3 is $1/3$ since each of the 3 points is equally likely to be the middle one.

18. X is uniformly distributed over $(0, L/2)$ and Y is uniformly distributed over $(L/2, L)$

$f(x,y) = f(x) f(y) = \frac{4}{L^2}$.

so $P(Y-X > L/3) = \iint_{y-x>L/3} \frac{4}{L^2}\,dy\,dx$ since $\begin{cases} y-x > L/3 \\ 0 < x < L/2 \\ \frac{L}{2} < y < L \end{cases}$ so case (i) $\begin{cases} 0 < x < \frac{L}{6} \\ \frac{L}{2} < y < L \end{cases}$ case (ii) $\begin{cases} \frac{L}{6} < x < \frac{L}{2} \\ \frac{L}{3} + x < y < L \end{cases}$

$$= \frac{4}{L^2}\left[\int_0^{L/6}\int_{L/2}^{L} dy\,dx + \int_{L/6}^{L/2}\int_{L/3+x}^{L} dy\,dx\right]$$

$$= \frac{4}{L^2}\left[\frac{L^2}{12} + \int_{\frac{L}{6}}^{\frac{L}{2}}\left[\frac{2}{3}L - x\right]dx\right] = \frac{1}{3} + \frac{4}{L^2}\left[\frac{2}{3}L\cdot\frac{L}{3} - \frac{1}{2}\left(\frac{L^2}{4} - \frac{L^2}{36}\right)\right] = \frac{7}{9}.$$

19. $f(x,y) = 1/x$, $0<y<x<1$ is a joint density function since

$$\iint_{0<y<x<1} f(x,y)\,dx\,dy = \int_0^1\int_y^1 \frac{1}{x}\,dx\,dy = \int_0^1\int_0^x \frac{1}{x}\,dy\,dx = 1.$$

(a) $\int_y^1 f(x,y)\,dx = \int_y^1 \frac{1}{x}\,dx = -\ln(y)$, $0<y<1$.

(b). $\int_0^x f(x,y)\,dy = \int_0^x \frac{1}{x}\,dy = 1$, $0 < \overset{x}{\cancel{y}} < 1$

(c). $EX = \int_0^1 x f(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}$

(d). $Ey = \int_0^1 y\,[-\ln(y)]\,dy$. Integrating by parts

$$= -\frac{y^2}{2}\ln(y)\Big|_0^1 + \int_0^1 \frac{y^2}{2}\cdot\frac{1}{y}\,dy = \int_0^1 \frac{y}{2}\,dy = \frac{1}{4}$$

20. (a) compute the marginal density of x, y

$$f(x) = \int_0^\infty f(x,y)\,dy = \int_0^\infty x e^{-(x+y)}\,dy = xe^{-x} \quad , \; x>0$$

$$f(y) = \int_0^\infty f(x,y)\,dx = \int_0^\infty xe^{-x-y}\,dx = e^{-y}\left[-xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx\right] = e^{-y}$$

$y>0$

then $f(x,y) = f(x)f(y)$, X and Y are independent

(b). $f(x) = \int_x^1 f(x,y)\,dy = 2.(1-x)$, $0<x<1$

$f(y) = \int_0^y f(x,y)\,dy = 2y \quad 0<y<1$

$f(x)f(y) = 4(1-x)y \neq f(x,y) = 2$. so X and Y are not independent.

24. N denote the number of trials needed to obtain an outcome that is not equal to 0

so N is from ~~Binomial~~ Geometric distribution with $p = 1-P_0$

(a) $P(N=n) = (1-P_0)\cdot P_0^{\,n-1}$.

(b). $P(X=j) = P_j/(1-P_0)$.

(c). $P(N=n, X=j) = P_0^{\,n-1}P_j = P(N=n)\,P(X=j)$

33. Let X denote Jill's score and Let Y denote Jack's score, $X \sim N(170, 20^2)$

$Y \sim N(160, 15^2)$

(a) $X - Y \sim N(10, 20^2+15^2)$

$$P(Y>X) = P(X-Y<-.5) = P\left(\frac{X-Y-10}{\sqrt{20^2+15^2}} \le \frac{-.5-10}{\sqrt{20^2+15^2}}\right) \approx P(Z \le -.42) = .3372.$$

(b) $X+Y \sim N(330, 15^2+20^2)$

$$P(X+Y>350) = P(X+Y>350.5) = P\left(\frac{X+Y-330}{\sqrt{15^2+20^2}} > \frac{350.5-330}{\sqrt{15^2+20^2}}\right) = P(Z>.82) \approx .2061$$

39. (a) $P(X=j, Y=i) = \frac{1}{5}\cdot\frac{1}{j} \quad j=1,\cdots,5 \quad i=1,\cdots,j$

(b) $$P(X=j \mid Y=i) = \frac{P(X=j, Y=i)}{\sum_{j=i}^{5} P(X=j, Y=i)} = \frac{\frac{1}{5j}}{\sum_{j=i}^{5}\frac{1}{5j}} = \frac{\frac{1}{j}}{\sum_{j=i}^{5}\frac{1}{j}}, \quad 5 \geq j \geq i$$

(c) if X and Y are independent, then $P(X=j \mid Y=i) = P(X=j) = \frac{1}{5}$

But this equation doesn't hold. so X and Y are not independent.

49. (a) $$P(\min X_i \leq a) = 1 - P(\min X_i > a) = 1 - \prod_{i=1}^{5} P(X_i > a) = 1 - \prod_{i=1}^{5}\int_a^\infty \lambda e^{-\lambda x}\,dx$$

$$= 1 - \prod_{i=1}^{5} e^{-\lambda a} = 1 - e^{-5\lambda a}$$

(b) $$P(\max X_i \leq a) = \prod_{i=1}^{5} P(X_i < a) = \prod_{i=1}^{5}\int_0^a \lambda e^{-\lambda x}\,dx = \prod_{i=1}^{5}(1-e^{-\lambda a}) = (1-e^{-\lambda a})^5$$

Theoretical Exercises :

11 $$I = P(X_1 < X_2 < X_3 < X_4 < X_5) = \int\!\!\int\!\!\int\!\!\int\!\!\int_{X_1<X_2<X_3<X_4<X_5} f(x_1)\cdots f(x_5)\,dx_1\cdots dx_5$$

Let $u_i = F(x_i)$

$$= \int_{F(X_1)<F(X_2)<\cdots<F(X_5)} \cdots \int dF(x_1)\cdots dF(x_5)$$

$$= \int_{u_1<u_2<u_3<u_4<u_5} \cdots \int du_1 \cdots du_5$$

$$= \int_0^1\int_0^{u_5}\int_0^{u_4}\int_0^{u_3}\int_0^{u_2} du_1\cdots du_5$$

$$= \int_0^1\int_0^{u_5}\int_0^{u_4}\int_0^{u_3} u_2\,du_2\,du_3\,du_4\,du_5$$

$$= \int_0^1\int_0^{u_5}\int_0^{u_4}\frac{u_3^2}{2}\,du_3\,du_4\,du_5$$

$$= \int_0^1\int_0^{u_5}\frac{u_4^3}{6}\,du_4\,du_5$$

$$= \int_0^1\frac{u_5^4}{24}\,du_5$$

$$= \frac{1}{120}$$ I does not depend on F.

15. Since X and Y are independent binomial random variables with n and p.

then $X+Y$ is binomial random variable with $2n$ and p.

then
$$p(X=k \mid X+Y=m) = \frac{p(X=k,\ X+Y=m)}{p(X+Y=m)} = \frac{p(X=k)\,p(Y=m-k)}{p(X+Y=m)}$$
$$= \frac{\binom{n}{k} p^k (1-p)^{n-k} \binom{n}{m-k} p^{m-k} (1-p)^{n-m+k}}{\binom{2n}{m} p^m (1-p)^{2n-m}}$$
$$= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$

22. $p([X]=n,\ X-[X]\le x) = p(n\le X<n+x) = \int_n^{n+x} \lambda e^{-\lambda x}\,dx = e^{-n\lambda} - e^{-(n+x)\lambda} = e^{-n\lambda}(1-e^{-x\lambda})$

$p([X]=n) = p(n\le X<n+1) = \int_n^{n+1} \lambda e^{-\lambda x}\,dx = e^{-n\lambda} - e^{-(n+1)\lambda} = e^{-n\lambda}(1-e^{-\lambda})$.

$$p(X-[X]\le x) = \sum_{n=0}^{\infty} p(X-[X]\le x \mid [X]=n)\,p([X]=n).$$
$$= \sum_{n=0}^{\infty} \cancel{p(X\le n+x)\,p([X]=n)}\ p(n\le X<n+x)$$
$$= \sum_{n=0}^{\infty} e^{-n\lambda}(1-e^{-x\lambda})$$
$$= (1-e^{-x\lambda})\frac{1}{1-e^{-\lambda}}.$$

$\therefore\ p([X]=n)\,p(X-[X]\le x) = p([X]=n,\ X-[X]\le x)$ they are independent

23. Let $Y=\max(X_1,\cdots,X_n)$, $Z=\min(X_1,\cdots,X_n)$.

$p(Y\le x) = \prod_{i=1}^{n} p(X_i\le x) = F^n(x)$.

$p(Z\le x) = 1-p(Z>x) = 1-\prod_{i=1}^{n} p(X_i>x) = 1-(1-F(x))^n.$