2009 Fall Stat 426 : Homework 1

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Announcement: The homework carries 50 points and contributes 5 points to the total grade. Your score on the homework is scaled down to out of 5 and recorded.

1. A geometric random variable W takes values $\{1,\ 2,\ 3,\ .\ .\ .\}$ and $P(W = j) = \theta(1-\theta)^{j-1}$,where $0 < \theta < 1$.

   (a) Prove that for any two positive integers i, j, it is the case that, $P(W > i + j | W > i) = P(W > j)$.

   (b) Indeed, the converse is also true. We show that if W is a discrete random variable taking values $\{1, 2, 3, .\ .\ .\}$ with probabilities $\{p_1, p_2, p_3, .\ .\ .\}$ and satisfies the memoryless property, then W must follow a geometric distribution.

   Follow these steps to establish the fact that W is geometric. Using the fact that W has the memoryless property, show that

   $$P(W > m) = (P(W > 1))^m,$$

   for any $m \geq 2$. As a first step towards proving this show that

   $$P(W > 2) = (P(W > 1))^2$$

   Define $\theta = P(W = 1)$ and $1 - \theta = P(X > 1)$. You now have,

   $$P(W > m) = (1-\theta)^m,$$

   for any $m \geq 2$. Use this to show that for any $m \geq 2$,

   $$P(W = m) = \theta(1-\theta)^{m-1}.$$

   But for $m = 1$,

   $$P(W = m) = P(W = 1) = \theta = \theta(1-\theta)^{m-1},$$

   trivially and the proof is complete. (5 + 7 = 12 points)

2. If X is random variable with distribution function F, with continuous non-vanishing density f, obtain the density function of the random variable $Y = X^k$, for an *even integer* k.

   Hint: Express the probability of the event $(X^k \leq y)$ in terms of the distribution function F of X and proceed from there. (6 points)

3. Let T be an exponential random variable with parameter β and let W be a random variable independent of T which assumes the value 1 with probability 2/3 and the value -1 with probability 1/3. Find the density of $X = WT$.

   Hint: It would help to split up the event $\{X \leq x\}$ as the union of $\{X \leq x, W = 1\}$ and $\{X \leq x, W = -1\}$. (7 points)

4. (a) A line is drawn through the point $(-1, 0)$ in a random direction. Let $(0, Y)$ denote the point at which it cuts the Y axis. Show that Y has the standard Cauchy density:

$$f_Y(y) = \frac{1}{\pi} \frac{1}{1+y^2}, y \in \mathbb{R}.$$

   (b) Let X and Y be i.i.d. $N(0, 1)$ random variables. Show that X/Y also has the standard Cauchy density. (7 + 7 =14 points)

5. Let X be an Exp(1) random variable. Let $[X]$ denote the largest integer not exceeding X. Show that

$$P([X] = m, X - [X] \leq t) = e^{-\lambda m}(1 - e^{-\lambda t}).$$

   Work out the marginals of $[X]$ and $X - [X]$ and deduce that these two random variables are independent. (6 + 5 = 11 points)