FINAL EXAM: STATISTICS 426

Moulinath Banerjee

University of Michigan

April 19, 2006

Announcement: The exam carries 70 points but the maximum you can score is 60. Half of your score contributes to your grade in the course.

(1) Let X_1 and X_2 be i.i.d. $N(0,1)$ random variables. Define $Y_1 = X_1$ and $Y_2 = X_1/X_2$. Compute the joint distribution of (Y_1, Y_2) and show that the marginal density of Y_2 is:

$$f_{Y_2}(y_2) = \frac{1}{\pi\,(1+y_2^2)}\ , \quad y_2 \in (-\infty, \infty)\,.$$

(10 points)

(2) Let X be a single random variable following $\text{Exp}(\lambda)$. Define $T(X) = 1$ if $X > 1$ and $T(X) = 0$ otherwise. Set $\psi(\lambda) = e^{-\lambda}$.

Show that $T(X)$ is unbiased for $\psi(\lambda)$ and find the information bound for unbiased estimators of $\psi(\lambda)$. Show that the variance of $T(X)$ is strictly larger than the information bound. (You may use the fact that $e^{\lambda} - 1 > \lambda^2$.) (10 points)

(3) Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. random variables following a geometric distribution. Thus, for each i, $P(X_i = x) = q^{x-1}p$ where x is a positive integer, and $0 < p, q < 1$ and $p + q = 1$.

Compute an explicit expression for the probability that the minimum of the X_i's is larger than a fixed integer x. What happens to this probability for fixed x (say $x = 1$) as n becomes large? What is the intuitive explanation behind this phenomenon? (10 points)

(4) A biologist is interested in measuring the ratio of mean weights of animals of two species. However, the species are extremely rare and after much effort she succeeds in measuring the weights of one animal from the first species and one from the second. Let X_1 and X_2 denote these weights. It is assumed that $X_1 \sim N(\theta_1, 1)$ and $X_2 \sim N(\theta_2, 1)$. Interest lies in estimating θ_1/θ_2.

Compute the distribution of

$$h(X_1, X_2, \theta_1, \theta_2) = \frac{\theta_2 X_1 - \theta_1 X_2}{\sqrt{\theta_1^2 + \theta_2^2}}$$

and conclude that

$$\frac{X_1 - (\theta_1/\theta_2) X_2}{\sqrt{(\theta_1/\theta_2)^2 + 1}}$$

is a pivot. Discuss how you can use this pivot to construct a confidence set for the ratio of mean weights. (10 points)

(5) Consider two particles situated at locations X_1 and X_2 on the horizontal axis and particles Y_1 and Y_2 situated at locations Y_1 and Y_2 on the vertical axis. It may be assumed that all random variables are independent. Furthermore X_1 and X_2 are i.i.d. $N(0, \sigma_1^2)$ and Y_1 and Y_2 are i.i.d. $N(0, \sigma_2^2)$. You only observe $X_1 - X_2$ and $Y_1 - Y_2$ based on which you want to estimate the ratio of standard deviations σ_1/σ_2.

(i) Show that

$$H(X_1, X_2, Y_1, Y_2, \sigma_1/\sigma_2) = \frac{| X_1 - X_2 | \sigma_2}{| Y_1 - Y_2 | \sigma_1}$$

is a pivot. What is its distribution?

(ii) If $\sigma_1 = \sigma_2 = \sigma$, calculate the distribution of

$$\frac{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}{2\,\sigma^2}$$

and indicate how you can construct a C.I. for σ^2 based on the above expression. $(5 + 5 = 10$ points)

(6) Let $X_1, X_2, \ldots, X_n$ be i.i.d. Uniform$(-\theta, \theta)$, where $\theta > 0$. Find a MOM and the MLE of θ. Is the MLE unbiased? (10 points)

(7) Consider i.i.d. observations $X_1, X_2, \ldots, X_n$ where each X_i follows a normal distribution with mean and variance both equal to $1/\theta$, where $\theta > 0$. Thus,

$$f(x, \theta) = \frac{\sqrt{\theta}}{\sqrt{2\pi}} \exp\left(-\frac{(x - \theta^{-1})^2}{2\theta^{-1}} \right).$$

Show that the MLE is one of the solutions to the equation:

$$\theta^2\, W - \theta - 1 = 0$$

where $W = n^{-1} \sum_{i=1}^n X_i^2$. Determine which root it is and compute its approximate variance in large samples.

(1) X_1, X_2 i.i.d $N(0,1)$ random variables.

$$Y_1 = X_1 \quad , \quad Y_2 = \frac{X_1}{X_2}.$$

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}(x_1^2+x_2^2)\right]$$

$$y_1 = x_1 \quad , \quad y_2 = \frac{x_1}{x_2}$$

$(y_1, y_2) \longleftrightarrow (x_1, x_2)$ is a 1-1 transformation

$$\Rightarrow \left.\begin{aligned} x_1 &= y_1 \\ x_2 &= \frac{x_1}{y_2} = \frac{y_1}{y_2} \end{aligned}\right\}$$

Next:

$$\begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{y^2} & -\frac{y_1}{y_2^2} \end{bmatrix} = \underset{(\text{say})}{M}$$

Then the Jacobian $J = \text{abs}(\det(M))$

$$= \frac{|y_1|}{y_2^2}.$$

Thus:

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}\left(y_1^2 + \frac{y_1^2}{y_2^2}\right)\right] \frac{|y_1|}{y_2^2}$$

Marginal density of Y_2:

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\, dy_1$$

$$f_{Y_2}(y_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}y_1^2 - \frac{1}{2y_2^2}y_1^2\right]\frac{|y_1|}{y_2^2}\,dy_1$$

$$= \frac{1}{2\pi y_2^2}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}y_1^2\left(1+\frac{1}{y_2^2}\right)\right]\frac{|y_1|}{1}\,dy_1$$

$$= \frac{2}{2\pi y_2^2}\int_{0}^{\infty} \exp\left[-\left(1+\frac{1}{y_2^2}\right)\frac{y_1^2}{2}\right] y_1\,dy_1$$

Set $w = \frac{y_1^2}{2}$, then $dw = y_1\,dy_1$, so

$$f_{Y_2}(y_2) = \frac{1}{\pi y_2^2}\int_0^\infty \exp\left[-\left(1+\frac{1}{y_2^2}\right)w\right]dw$$

$$= \frac{1}{\pi y_2^2}\cdot\frac{1}{\left(1+\frac{1}{y_2^2}\right)}\int_0^\infty \left(1+\frac{1}{y_2^2}\right)\exp\left[-\left(1+\frac{1}{y_2^2}\right)w\right]dw$$

But $\int_0^\infty \left(1+\frac{1}{y_2^2}\right)\exp\left[-\left(1+\frac{1}{y_2^2}\right)w\right]dw = 1$

(Why?)

$$\therefore f_{Y_2}(y_2) = \frac{1}{\pi y_2^2}\,\frac{y_2^2}{1+y_2^2}, \quad y_2 \in (-\infty, \infty)$$

$$= \frac{1}{\pi(1+y_2^2)}, \quad y_2 \in (-\infty, \infty)$$

which is the Cauchy density

Solution: Final Exam, Statistics 426

2. $X \sim \exp(\lambda)$.

$T(X) = 1(X>1)$.

$\psi(\lambda) = e^{-\lambda}$.

$E_\lambda(T(X)) = E_\lambda[1(X>1)] = 1 \cdot P_\lambda(X>1) = e^{-\lambda}$.

So $T(X)$ is unbiased for $\psi(\lambda)$.

For any unbiased estimator $S(X)$ of $e^{-\lambda}$,

we have:

$$Var_\lambda(S(X)) \geq \frac{\psi'(\lambda)^2}{I(\lambda)} = e^{-2\lambda}\lambda^2.$$

since $\psi'(\lambda) = -e^{-\lambda}$

and $$I(\lambda) = E_\lambda\left[-\frac{\partial^2}{\partial\lambda^2}\log(\lambda \cdot e^{-\lambda X_1})\right]$$

$(X \equiv X_1)$

$X_1 \sim \exp(\lambda)$

$$= E_\lambda\left[-\frac{\partial^2}{\partial\lambda^2}[\log\lambda - \lambda X_1]\right]$$

$$= E_\lambda\left[-\frac{\partial}{\partial\lambda}\left[\frac{1}{\lambda} - X_1\right]\right]$$

$$= \frac{1}{\lambda^2}$$

$Var_\lambda(T(X)) = e^{-\lambda}(1-e^{-\lambda})$.

To show: $e^{-\lambda}(1-e^{-\lambda}) > e^{-2\lambda}\lambda^2$

i.e $e^{2\lambda}e^{-\lambda}(1-e^{-\lambda}) > \lambda^2$

i.e $e^{\lambda} - 1 > \lambda^2$, which we are given!

3. $P(\min_{1 \le i \le n} X_i > x)$

$= P(X_1 > x, X_2 > x, \ldots, X_n > x)$

$= P(X_1 > x) \cdot P(X_2 > x) \cdots P(X_n > x)$

$= P(X_1 > x)^n$

But $P(X_1 > x) = q^x$, so $P(\min_{1 \le i \le n} X_i > x)$

$= q^{nx} \longrightarrow 0$ as $n \to \infty$

The intuitive explanation here is that as more and more variables are observed the chance of seeing at least one 1 (which is the minimum possible value) i.e. a success right at the first stage keeps increasing.

4. $X_1 \sim N(\theta_1; 1)$, $X_2 \sim N(\theta_2, 1)$.

$$h(X_1, X_2, \theta_1, \theta_2) = \frac{\theta_2 X_1 - \theta_1 X_2}{\sqrt{\theta_1^2 + \theta_2^2}}$$

$$\theta_2 X_1 - \theta_1 X_2 \sim N(0, \theta_1^2 + \theta_2^2)$$

since $E(\theta_2 X_1 - \theta_1 X_2) = \theta_2 E(X_1) - \theta_1 E X_2$

$$= \theta_2 \theta_1 - \theta_1 \theta_2 = 0$$

$$Var(\theta_2 X_1 - \theta_1 X_2) = Var(\theta_2 X_1) + Var(\theta_1 X_2)$$

$$= \theta_2^2 Var(X_1) + \theta_1^2 Var(X_2)$$

$$= \theta_2^2 + \theta_1^2.$$

But $\dfrac{\theta_2 X_1 - \theta_1 X_2}{\sqrt{\theta_1^2 + \theta_2^2}} = \dfrac{X_1 - \frac{\theta_1}{\theta_2} X_2}{\sqrt{\frac{\theta_1^2 + \theta_2^2}{\theta_2^2}}}$

$$= \frac{X_1 - \frac{\theta_1}{\theta_2} X_2}{\sqrt{1 + \frac{\theta_1^2}{\theta_2^2}}}$$

Now, $\dfrac{\theta_2 X_1 - \theta_1 X_2}{\sqrt{\theta_1^2 + \theta_2^2}} \sim N(0,1)$

so $(X_1 - (\theta_1/\theta_2) X_2) \Big/ \sqrt{1 + \frac{\theta_1^2}{\theta_2^2}}$ is indeed a pivot.

To get a C.I for $\frac{\theta_1}{\theta_2} \equiv \eta$, say,

$$P\left[-z_{\alpha/2} \le \frac{X_1 - \eta X_2}{\sqrt{1+\eta^2}} \le z_{\alpha/2}\right] = 1-\alpha$$

i.e $$P\left[\frac{|X_1 - \eta X_2|}{\sqrt{1+\eta^2}} \le z_{\alpha/2}\right] = 1-\alpha$$

i.e $$P\left[(X_1 - \eta X_2)^2 - (1+\eta^2) z_{\alpha/2}^2 \le 0\right] = 1-\alpha$$

Thus: $\{\eta : (X_1 - \eta X_2)^2 - z_{\alpha/2}^2 (1+\eta^2) \le 0\}$ gives a level $1-\alpha$ confidence set for η. This can be expressed explicitly in terms of the roots of the quadratic equation involved.

5. (i) $$H(X_1, X_2, Y_1, Y_2, \sigma_1/\sigma_2) = \frac{|X_1 - X_2|\sigma_2}{|Y_1 - Y_2|\sigma_1}$$

To show this is a pivot,

consider $$H^2 = \frac{(X_1 - X_2)^2/\sigma_1^2}{(Y_1 - Y_2)^2/\sigma_2^2} = \frac{(X_1 - X_2)^2/2\sigma_1^2}{(Y_1 - Y_2)^2/2\sigma_2^2}$$

$$\left.\begin{aligned} X_1 - X_2 &\sim N(0, 2\sigma_1^2) \\ Y_1 - Y_2 &\sim N(0, 2\sigma_2^2) \end{aligned}\right\}$$ Also $X_1 - X_2$ and $Y_1 - Y_2$ are independent

$\therefore \dfrac{(X_1 - X_2)^2}{2\sigma_1^2} \sim \chi^2_1$ independently of

$$\frac{(Y_1 - Y_2)^2}{2\sigma_2^2} \sim \chi^2_1 .$$

It follows that $H^2 \sim F_{1,1}$.

and $\therefore H$ is distributed as $\sqrt{F_{1,1}}$.

(ii) If $\sigma_1 = \sigma_2 = \sigma$

note that $\dfrac{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}{2\sigma^2} \sim \chi^2_2$

and is therefore a pivot.

If $q_{\alpha/2}$ and $q_{1-\frac{\alpha}{2}}$ denote the appropriate quantiles of a χ^2_2 distribution (these can be explicitly determined), then:

$$\left\{ \sigma^2 : q_{\frac{\alpha}{2}} \le \frac{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}{2\sigma^2} \le q_{1-\frac{\alpha}{2}} \right\}$$

gives a level $1-\alpha$ C.I. This can be written down explicitly.

(6) Follow similar steps as in the solution to the last problem on HW 3.

$$L(\theta \mid X_1, \ldots, X_n) = \prod_{i=1}^{n} \frac{1}{2\theta} \mathbb{1}(-\theta \le X_i \le \theta)$$

$$= \frac{1}{2^n \theta^n} \prod_{i=1}^{n} \mathbb{1}(-\theta \le X_i \le \theta)$$

$$= \frac{1}{2^n \theta^n} \prod_{i=1}^{n} \mathbb{1}(|X_i| \le \theta)$$

$$= \frac{1}{2^n \theta^n} \mathbb{1}(\max_i |X_i| \le \theta).$$

Check that $\hat{\theta}_{MLE} = \max_i |X_i|$

To find a MOM,

note $\mu_1 = 0$,

$\mu_2(\theta) = E_\theta(X_1^2) = \frac{\theta^2}{3}$, since $|X_1| \sim \text{Unif}(0, \theta)$

and a MOM estimate is readily computable based on this formula.

(7) $X_1, X_2, \ldots, X_n$ i.i.d $N\left(\frac{1}{\theta}, \frac{1}{\theta}\right)$. $\quad \mu = \frac{1}{\theta}$

$\sigma^2 = \frac{1}{\theta}$.

$$f(x,\theta) = \frac{\sqrt{\theta}}{\sqrt{2\pi}} \exp\left(-\frac{\theta}{2}\left(x - \frac{1}{\theta}\right)^2\right)$$

$$L(\theta \mid x_1, .., x_n) = \prod_{i=1}^{n} \frac{\theta^{1/2}}{2\pi^{1/2}} \exp\left[-\frac{\theta}{2}\left(x_i - \frac{1}{\theta}\right)^2\right]$$

$$= \frac{\theta^{n/2}}{2\pi^{n/2}} \exp\left[-\frac{\theta}{2}\sum_{i=1}^{n}\left(x_i - \frac{1}{\theta}\right)^2\right]$$

$$\log L(\theta \mid x_1, .., x_n) = \text{constant} + \frac{n}{2}\log\theta - \frac{\theta}{2}\left[\sum_{i=1}^{n} x_i^2 - \frac{2}{\theta} n\bar{x} + \frac{n}{\theta^2}\right]$$

$$= \text{constant} + \frac{n}{2}\log\theta - \frac{\theta}{2}\sum x_i^2 + n\bar{x} - \frac{n}{2\theta}$$

$$\frac{\partial \log L(\theta \mid x_1, .., x_n)}{\partial\theta} = \frac{n}{2\theta} - \frac{1}{2}\sum x_i^2 + \frac{n}{2\theta^2}$$

Setting this to be 0 gives:

$$\frac{1}{2\theta} - \frac{1}{2}\overline{x^2} + \frac{1}{2\theta^2} = 0$$

$$\Rightarrow \theta - \theta^2\overline{x^2} + 1 = 0$$

i.e $\theta^2\overline{x^2} - \theta - 1 = 0$

The two roots are given by $\dfrac{1 \pm \sqrt{1+4\overline{X^2}}}{2\overline{X^2}}$

and the admissible root is

$$\hat{\theta}_{MLE} = \frac{1+\sqrt{1+4\overline{X^2}}}{2\overline{X^2}}$$

$$\hat{\theta}_{MLE} \underset{appx}{\approx} N\left(0, \frac{1}{nI(\theta)}\right)$$

where $I(\theta) = -E_\theta\left[\frac{\partial^2}{\partial\theta^2} \log f(X_1, \theta)\right]$

$$= -E_\theta\left[\frac{\partial^2}{\partial\theta^2} \log \frac{\theta^{1/2}}{2\pi^{1/2}} \exp\left[-\frac{\theta}{2}\left(X_1 - \frac{1}{\theta}\right)^2\right]\right]$$

<u>Consider:</u> $\frac{\partial^2}{\partial\theta^2} \log \frac{\theta^{1/2}}{2\pi^{1/2}} \exp\left[-\frac{\theta}{2}\left(X_1 - \frac{1}{\theta}\right)^2\right]$

$$= \frac{\partial^2}{\partial\theta^2}\left[\frac{1}{2}\log\theta - \frac{\theta}{2}X_1^2 + X_1 - \frac{1}{2\theta}\right]$$

$$= \frac{\partial}{\partial\theta}\left[\frac{1}{2\theta} - \frac{1}{2}X_1^2 + \frac{1}{2\theta^2}\right]$$

$$= -\frac{1}{2\theta^2} - \frac{1}{\theta^3}.$$

So $I(\theta) = \frac{1}{2\theta^2} + \frac{1}{\theta^3}.$