HW1 Solution

Ch2 (P_{58} - P_{60})

24. $i=2$, (1,1) $p(i=2)=\frac{1}{36}$

$i=3$, (1,2), (2,1) $p(i=3)=\frac{2}{36}=\frac{1}{18}$

$i=4$, (1,3), (2,2), (3,1) $p(i=4)=\frac{3}{36}=\frac{1}{12}$

$i=5$, (1,4), (2,3), (3,2), (4,1) $p(i=5)=\frac{4}{36}=\frac{1}{9}$

$i=6$, (1,5), (2,4), (3,3), (4,2), (5,1) $p(i=6)=\frac{5}{36}$

$i=7$, (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) $p(i=7)=\frac{6}{36}=\frac{1}{6}$

$i=8$, (2,6), (3,5), (4,4), (5,3), (6,2) $p(i=8)=\frac{5}{36}$

$i=9$, (3,6), (4,5), (5,4), (6,3) $p(i=9)=\frac{4}{36}=\frac{1}{9}$

$i=10$, (4,6), (5,5), (6,4) $p(i=10)=\frac{3}{36}=\frac{1}{12}$

$i=11$, (5,6), (6,5) $p(i=11)=\frac{2}{36}=\frac{1}{18}$

$i=12$, (6,6) $p(i=12)=\frac{1}{36}$

total points 6

25. 5 can be rolled in 4 ways (1,4), (2,3), (3,2), (4,1)

7 can be rolled in 6 ways (1,6), (2,5), (3,4), (4,3), (5,2), (6,1)

E_n = events that 5 occurs on the nth roll and no 5 or 7 occurs on the first n-1 rolls

$$p(E_n)=\left(\frac{36-6-4}{36}\right)^{n-1}\cdot\frac{4}{36}=\left(\frac{26}{36}\right)^{n-1}\cdot\frac{4}{36}$$

$\sum_{n=1}^{\infty} p(E_n)$ is the desired probability

$$\sum_{n=1}^{\infty} p(E_n)=\frac{4}{36}\cdot\sum_{n=1}^{\infty}\left(\frac{26}{36}\right)^{n-1}=\frac{4}{36}\ \frac{1}{1-\frac{26}{36}}=\frac{2}{5}.$$

total points 8

41. p(6 comes up at least once) = 1 − p(6 doesn't come up in the 4 rolls)

$$=1-\frac{5^4}{6^4}$$

total points 4

47. every stranger has their birthday on different months.

$$p=\frac{12!}{12^{12}}$$

total points 4

51. First choose the m balls from the 5 balls ~~to~~ that will fall in the first compartment is $\binom{n}{m}$ choices. then the other $n-m$ balls are randomly distributed in $N-1$ compartments is $(N-1)^{n-m}$ choices

so p(m balls fall in the first compartment) $= \dfrac{\binom{n}{m}(N-1)^{n-m}}{N^m}$.

total points 6

53. Let A_i be the event that couple i sit next to each other.

Then $1 - p(\bigcup_{i=1}^{4} A_i)$ is the desired probability.

$$p(\bigcup_{i=1}^{4} A_i) = \sum_{i=1}^{4} p(A_i) - \sum_{i_1<i_2} p(A_{i_1}A_{i_2}) + \sum_{i_1<i_2<i_3} p(A_{i_1}A_{i_2}A_{i_3}) - p(A_1A_2A_3A_4)$$

$$p(A_i) = \frac{2\cdot 7!}{8!} \qquad p(A_{i_1}A_{i_2}) = \frac{2^2\cdot 6!}{8!} \qquad p(A_{i_1}A_{i_2}A_{i_3}) = \frac{2^3\cdot 5!}{8!}$$

$$p(A_1A_2A_3A_4) = \frac{2^4 4!}{8!}$$

so the desired probability $= 1 - 4\cdot\dfrac{2\cdot 7!}{8!} + 6\cdot\dfrac{2^2\cdot 6!}{8!} - 4\cdot\dfrac{2^3\cdot 5!}{8!} + \dfrac{2^4\cdot 4!}{8!}$

total points 8

$P_{61} - P_{63}$

5. Define $F_1 = E_1$, $F_2 = E_2 \cap E_1^c$ .... $F_i = E_i \bigcap_{j=1}^{i-1} E_j^c$. total points 6

11. $1 \geq p(E\cup F) = p(E) + p(F) - p(EF)$

so $p(E) + p(F) - 1 \leq p(EF)$.

Let $p(E) = .9$ $F(F) = .8$ $p(EF) \geq .9 + .8 - 1 = .7$ total points 6

12. because. $p(EF) + p(EF^c) = p(E)$. $p(EF) + p(E^cF) = p(F)$.

so $p(EF^c \cup E^cF) = p(EF^c) + p(E^cF)$ $\because$ EF^c and E^cF are disjoint sets

$= (p(E) - p(EF)) + (p(F) - p(EF))$

$= p(E) + p(F) - 2p(EF)$.

total points 6

16. $p(E_1E_2\cdots E_n) \geq p(E_1E_2\cdots E_{n-1}) + p(E_n) - 1$ by Bonferonni's Inequality

$$\geq p(E_1\cdots E_{n-2}) + p(E_{n-1}) + p(E_n) - 2.$$

$$\vdots \qquad \text{by induction hypothesis}$$

$$\geq \sum_{i=1}^{n} p(E_i) - (n-1).$$

total points 6

19. A total of k balls will be withdrawn if there are $r-1$ red balls in the first $k-1$ withdrawals and the kth withdrawl is a red ball.

then the probability that the first $k-1$ withdrawls has $r-1$ red balls is. $\dfrac{\binom{n}{r-1}\binom{m}{k-r}}{\binom{n+m}{k-1}}$ and the probability that the kth withdrawl is a red ball is $\dfrac{n-r+1}{n+m-k+1}$ so $p = \dfrac{\binom{n}{r-1}\binom{m}{k-r}(n-r+1)}{\binom{n+m}{k-1}(n+m-k+1)}$

total points 8

20. The experiment would be flipping a coin until you toss a head. Like H, TH, TTH, TTTH, ... so the probability of the nth time you toss a head is $\frac{1}{2^n}$, this experiment whose sample space consists of a countably infinite number of points and all points have positive probability which is $\frac{1}{2^n}$, $n=1,2\cdots$ But not all points can be equally likely, Or the probabilities would not be able to add up to 1.

total points 6.

P64 ~ P65

6. Let R be the events that both balls are red, Let B be the events that they are both black. $p(R) = \dfrac{3\cdot 4}{6\cdot 10}$ $\quad p(B) = \dfrac{3\cdot 6}{6\cdot 10}$

$p(R) + p(B) = 1/2.$

total points 6

9. Let $S = \bigcup_{i=1}^{n} A_i$, and consider the experiment of randomly choosing an experiment of S. Then $p(A) = N(A) / N(S)$. From proposition 4.4.

$$p(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} p(A_i) - \sum_{i_1 < i_2} p(A_{i_1} A_{i_2}) + \cdots + (-1)^{r+1} \sum_{i_1 < i_2 < \cdots < i_r} p(A_{i_1} A_{i_2} \cdots A_{i_r})$$

$$+ \cdots + (-1)^{n+1} p(A_1 A_2 \cdots A_n)$$ Let $S = A_1 \cup A_2 \cup \cdots \cup A_n$.

$$\frac{N(A_1 \cup A_2 \cup \cdots \cup A_n)}{N(A_1 \cup A_2 \cup \cdots \cup A_n)} = \sum_{i=1}^{n} \frac{N(A_i)}{N(A_1 \cup A_2 \cup \cdots \cup A_n)} - \sum_{i_1 < i_2} \frac{N(A_{i_1} A_{i_2})}{N(A_1 \cup A_2 \cup \cdots \cup A_n)} + \cdots +$$

$$(-1)^{r+1} \sum_{i_1 < i_2 < \cdots < i_r} \frac{N(A_{i_1} A_{i_2} \cdots A_{i_r})}{N(A_1 \cup A_2 \cup \cdots \cup A_n)} + \cdots + (-1)^{n+1} \frac{N(A_1 A_2 \cdots A_n)}{N(A_1 \cup A_2 \cup \cdots \cup A_n)}$$

total points 8

$$\therefore \; N\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} N(A_i) - \sum_{i<j} N(A_i A_j) + \cdots + (-1)^{n+1} N(A_1 \cdots A_n)$$

14. Let $B_1 = A_1$, $B_i = A_i \left(\bigcup_{j=1}^{i-1} A_j\right)^c$, $i > 1$. Then $p\left(\bigcup_{i=1}^{\infty} A_i\right) = p\left(\bigcup_{i=1}^{\infty} B_i\right)$

and B_i are disjoint events so $p\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} p(B_i) \le \sum_{i=1}^{\infty} p(A_i)$

so $p\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} p(A_i)$

total points 6

15. $$p\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - p\left(\left(\bigcap_{i=1}^{\infty} A_i\right)^c\right) = 1 - p\left(\bigcup_{i=1}^{\infty} A_i^c\right) \ge 1 - \sum_{i=1}^{\infty} p(A_i^c) = 1$$

result from 14. problem.

total points 6

whole points 100.