Midterm 2: Stat 426.

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Announcement: The total number of points is 33 but the maximum you can score is 30. You have 1 hour.

A potentially relevant result: The sum of the first n natural numbers i.e. $1 + 2 + \ldots + n$ is $n\,(n+1)/2$.

Problem 1: Consider $X_1, X_2, \ldots, X_n$ which are i.i.d. $N(0, \sigma^2)$, where σ^2 is unknown. Thus, the common density is,

$$f(x, \sigma^2) = \frac{1}{\sqrt{2\,\pi}\,\sigma} \exp\left(-\frac{x^2}{2\,\sigma^2}\right).$$

- (i) Write down the joint likelihood for the data.
- (ii) Compute the MLE of σ^2 and show that this is the same as a Method of Moments estimator of σ^2.

(3 + 5 = 8 points)

Problem 2: A company has manufactured certain objects and has printed a serial number on each manufactured object. The serial numbers start at 1 and end at N, where N is the number of objects manufactured. The problem is to estimate N (the parameter). A simple random sample of size n is drawn *with* replacement from the lot; let $X_1, X_2, \ldots, X_n$ denote the numbers on the objects that come up. Thus $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with common frequency function $p(x, N) = P_N(X_1 = x) = 1/N$, with x taking integer values from 1 through N.

- (i) Compute $\mu_1 = E(X_1)$ and express N as a function of μ_1. Obtain a Method of Moments estimate of N. (Keep in mind that an estimate of N has to be an integer, so you might need to tweak your Method of Moments estimate a bit, to take this fact into account.)
- (ii) Write down the joint likelihood function for the data and compute $\hat{N}$, the MLE of N.
- (iii) For any integer $M < N$ show that the probability that $\hat{N} > M$ converges to 1 with increasing sample size n.

(4 + 5 + 3 = 12 points)

Problem 3: Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with common density function,

$$f(x, \theta) = (\theta + 1)\, x^{\theta} \quad 0 \le x \le 1\,.$$

(a) Find a MOM estimate of θ. (Hint: Compute $\mu_1 = E(X_1)$.)

(b) Find the MLE of θ. (3 + 3 = 6 points)

Problem 4: Let $X_1, X_2, \ldots, X_n$ be i.i.d. exponential(λ). Show that $2\,\lambda\, n\,\overline{X}$ follows a χ^2_{2n} distribution and use this result to find a level $1 - \alpha$ confidence interval for λ. (Hint: What is the distribution of $2\,\lambda\, X_1$?) (7 points)