Midterm 2 solutions.

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Announcement: The total number of points is 33 but the maximum you can score is 30. You have 1 hour.

A potentially relevant result: The sum of the first n natural numbers i.e. $1 + 2 + \ldots + n$ is $n\,(n+1)/2$.

Problem 1: Consider $X_1, X_2, \ldots, X_n$ which are i.i.d. $N(0, \sigma^2)$, where σ^2 is unknown. Thus, the common density is,

$$f(x, \sigma^2) = \frac{1}{\sqrt{2\,\pi}\,\sigma} \exp\left(-\frac{x^2}{2\,\sigma^2}\right) .$$

- (i) Write down the joint likelihood for the data.
- (ii) Compute the MLE of σ^2 and show that this is the same as a Method of Moments estimator of σ^2.

Solution: You should be able to derive this completely from the notes on "Methods of Estimation" posted on the web. See Example 4 in the section on Maximum Likelihood Estimation. Set $\mu = 0$ in the joint likelihood (to start out with) and maximize the resulting likelihood just with respect to σ or σ^2. You should get,

$$\hat{\sigma}^2_{MLE} = \frac{1}{n} \sum_{i=1}^{n} X_i^2 .$$

Problem 2: A company has manufactured certain objects and has printed a serial number on each manufactured object. The serial numbers start at 1 and end at N, where N is the number of objects manufactured. The problem is to estimate N (the parameter). A simple random sample of size n is drawn *with* replacement from the lot; let $X_1, X_2, \ldots, X_n$ denote the numbers on the objects that come up. Thus $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with common frequency function $p(x, N) = P_N(X_1 = x) = 1/N$, with x taking integer values from 1 through N.

- (i) Compute $\mu_1 = E(X_1)$ and express N as a function of μ_1. Obtain a Method of Moments estimate of N. (Keep in mind that an estimate of N has to be an integer, so you might need

to tweak your Method of Moments estimate a bit, to take this fact into account.)
Solution:

$$E(X_1) = \sum_{j=1}^{N} j\, P(X_1 = j) = \sum_{j=1}^{N} j\, \frac{1}{N} = \frac{N(N+1)}{2N} = \frac{N+1}{2}\,.$$

Thus,

$$N = 2\,\mu_1 - 1\,.$$

Thus

$$\hat{N}_{MOM} = 2\,\overline{X} - 1\,.$$

Since N is an integer, the estimate above should be rounded off to the nearest integer.

- (ii) Write down the joint likelihood function for the data and compute $\hat{N}$, the MLE of N.
Solution:

$$L(X, N) = \Pi_{i=1}^{n} f(X_i, N) = \Pi_{i=1}^{n} \frac{1}{N}\, I(X_i \leq N) = \frac{1}{N^n}\, I(X_{(n)} \leq N)\,.$$

Recall that $X_{(n)}$ denotes the n'th ordered statistic, or the maximum. As a function of N, $L(X, N) = 0$ for $N < X_{(n)}$ (since $I(X_{(n)} \leq N) = 0$) and for $N \geq X_{(n)}$ it is simply $1/N^n$, showing that the maximum is attained at $N = X_{(n)}$. This is therefore the MLE.

Problem 3: Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with common density function,

$$f(x, \theta) = (\theta + 1)\, x^{\theta} \quad 0 \leq x \leq 1\,.$$

(a) Find a MOM estimate of θ. (Hint: Compute $\mu_1 = E(X_1)$.)
Solution:

$$\begin{aligned} E(X_1) &= \int_0^1 (\theta + 1)\, x^{\theta+1}\, dx \\ &= \frac{\theta + 1}{\theta + 2}\,. \end{aligned}$$

Thus,

$$\mu_1 = \frac{\theta + 1}{\theta + 2}\,,$$

which gives,

$$\theta = \frac{1 - 2\,\mu_1}{\mu_1 - 1}\,.$$

Thus,

$$\hat{\theta}_{MOM} = \frac{1 - 2\,\overline{X}}{\overline{X} - 1}\,.$$

(b) Find the MLE of θ.
Solution: The likelihood function is,

$$L(X,\theta) = \Pi_{i=1}^{n} f(X_i,\theta) = (\theta+1)^n \, \Pi_{i=1}^{n} X_i^{\theta}\,.$$

Thus,

$$\log\, L(X,\theta) = n\, \log(\theta+1) + \theta \sum_{i=1}^{n} \log(X_i)\,.$$

Setting,

$$\frac{\partial}{\partial\,\theta} \log\, L(X,\theta) = 0\,,$$

gives,

$$\frac{n}{\theta+1} + \sum_{i=1}^{n} \log(X_i) = 0\,.$$

Solving for θ we get,

$$\hat{\theta}_{MLE} = -\frac{n}{\sum_{i=1}^{n} \log(X_i)} - 1\,.$$

Problem 4: Let $X_1, X_2, \ldots, X_n$ be i.i.d. exponential(λ). Show that $2\,\lambda\, n\,\overline{X}$ follows a χ^2_{2n} distribution and use this result to find a level $1-\alpha$ confidence interval for λ. (Hint: What is the distribution of $2\,\lambda\, X_1$?) (7 points)
Solution: We have,

$$2\,\lambda\, n\,\overline{X} = \sum_{i=1}^{n} 2\,\lambda\, X_i\,.$$

Since X_i follows exp(λ), $2\,\lambda\, X_i$ follows an exponential(1/2) which is simply χ^2_2(from a Homework exercise). Since the X_i's are independent so are the $2\,\lambda\, X_i$'s and since the sum of n independent χ^2_2's is χ^2_{2n}, it follows that $2\,\lambda\, n\,\overline{X}$ follows χ^2_{2n} and is therefore a pivot.

Letting a_n denote the $\alpha/2$'th quantile of χ^2_{2n} and b_n denote the $1-\alpha/2$'th quantile of χ^2_{2n}, a level $1-\alpha$ confidence interval for λ is

$$\left\{\lambda:\; a_n \le 2\,\lambda\, n\,\overline{X} \le b_n\right\}\,.$$

This is simply the interval

$$\left[\frac{a_n}{2\,n\,\overline{X}}, \frac{b_n}{2\,n\,\overline{X}}\right]\,.$$