33. Let A be the event that a letter taken at random from his spelling is a vowel

Let B be the event that the writer is an Englishman.

We want to calculate $p(B|A)$

$$p(B|A) = \frac{p(AB)}{p(A)} = \frac{p(AB)}{p(A|B)p(B)+p(A|B^c)p(B^c)} = \frac{p(A|B)p(B)}{p(A|B)p(B)+p(A|B^c)p(B^c)}$$

if the men is English, $p(A|B) = \frac{1}{2}$ if the men is American, $p(A|B^c) = \frac{2}{5}$

$$\therefore p(B|A) = \frac{\frac{1}{2}\times 0.4}{\frac{1}{2}\times 0.4 + \frac{2}{5}\times 0.6} = \frac{5}{11}$$

37. Let B be the event that it is the fair coin. and Let A_1 be the event that first flip it shows head, A_2 be the event that on first and second flips, it shows head and A_3 be the event that all the ~~three~~ third flips it shows ~~head~~ tails

(a) $$p(B|A_1) = \frac{p(BA_1)}{p(A_1)} = \frac{p(A_1|B)p(B)}{p(A_1|B)p(B)+p(A_1|B^c)p(B^c)} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2}+1\cdot\frac{1}{2}} = \frac{1}{3}$$

(b) $$p(B|A_2) = \frac{p(A_2|B)p(B)}{p(A_2|B)p(B)+p(A_2|B^c)p(B^c)}$$

and $p(A_2|B)$ equals the probability that if the coin is fair, ~~the~~ it shows head twice.

so $p(A_2|B) = \frac{1}{4}$ and if the coin is two-headed, then it always shows head, $p(A_2|B^c) = 1$

so $$p(B|A_2) = \frac{\frac{1}{4}\cdot\frac{1}{2}}{\frac{1}{4}\cdot\frac{1}{2}+1\cdot\frac{1}{2}} = \frac{1}{5}$$

(c) if it shows tails. then the coin must be fair so $p(B|A_3) = 1$

45. Let B_i be the event that it was the ~~fifth~~ ith coin and A be the event that the coin shows head

$$p(B_5|A) = \frac{p(A|B_5)p(B_5)}{\sum_{i=1}^{10} p(A|B_i)p(B_i)} = \frac{5/10\cdot 1/10}{\sum_{i=1}^{10}\frac{i}{10}\cdot\frac{1}{10}} = \frac{\frac{5}{100}}{\frac{1}{100}\sum_{i=1}^{10} i} = \frac{\frac{5}{100}}{\frac{1}{100}\cdot\frac{10(10+1)}{2}} = \frac{5}{55} = \frac{1}{11}$$

47. Let B be the event that all the balls selected are white and A_i be the event that the die landed on i. because there are only five ~~balls~~ white balls in an urn, so $i=1,2,3,4,5$

$$p(B) = \sum_{i=1}^{5} p(B|A_i)p(A_i) \qquad p(A_i) = \frac{1}{6} \qquad p(B|A_i) = \frac{\binom{5}{i}}{\binom{15}{i}}$$

so $$p(B) = \frac{1}{6}\left[\frac{5}{15} + \frac{5}{15}\cdot\frac{4}{14} + \frac{5}{15}\cdot\frac{4}{14}\cdot\frac{3}{13} + \frac{5}{15}\cdot\frac{4}{14}\cdot\frac{3}{13}\cdot\frac{2}{12} + \frac{5}{15}\cdot\frac{4}{14}\cdot\frac{3}{13}\cdot\frac{2}{12}\cdot\frac{1}{11}\right] = 0.076$$

$$p(A_3|B) = \frac{p(B|A_3)p(A_3)}{p(B)} = \frac{\frac{1}{6}\cdot\frac{5}{15}\cdot\frac{4}{14}\cdot\frac{3}{13}}{p(B)} = 0.048.$$

49.

49. Let B be the event that the man has the cancer and A be the event that the test indicated an elevated PSA Level

(a). $p(B|A) = \dfrac{p(A|B)p(B)}{p(A|B)p(B)+p(A|B^c)p(B^c)}$ $\quad p(A|B) = .268$, $p(A|B^c) = .135$

case (i) $p(B) = .7$ then $p(B|A) = \dfrac{0.268 \times 0.7}{0.268\times0.7 + 0.135\times0.3} = 0.822.$

Case (ii) $p(B) = .3$ then $p(B|A) = \dfrac{0.268 \times 0.3}{0.268\times0.3 + 0.135\times0.7} = 0.460.$

(b) $p(B|A^c) = \dfrac{p(A^c|B)p(B)}{p(A^c|B)p(B)+p(A^c|B^c)p(B^c)}$ $\quad p(A^c|B) = 1 - p(A|B) = .732$ $\quad p(A^c|B^c) = 1 - .135 = .865$

case (i) $p(B) = 0.7$ $\quad p(B|A^c) = \dfrac{0.732\times0.7}{0.732\times0.7 + 0.865\times0.3} = 0.664$

case (ii) $p(B) = 0.3$ $\quad p(B|A^c) = \dfrac{0.732\times0.3}{0.732\times0.3+0.865\times0.7} = 0.266$

58. (a) All we knows when the procedure ends is that the two most flips were either H,T or T,H

Thus. p(the result is head) = p(the two most flips were H,T | the procedure ends)

$= p(H,T \mid H,T \text{ or } T,H) = \dfrac{p(H,T)}{p(H,T)+p(T,H)} = \dfrac{p(1-p)}{p(1-p)+(1-p)p} = \dfrac{1}{2}$

(b). Let A_n be the event that the nth flip is head, so the first $n-1$ flips are tails

so $p(A_n) = p\cdot(1-p)^{n-1}$ $\quad$ Let A be the event that the result is head.

$p(A) = \sum_{n=2}^{\infty} p(A_n) = \sum_{n=2}^{\infty} p(1-p)^{n-1} = p\cdot\dfrac{1-p}{1-(1-p)} = 1-p \neq \dfrac{1}{2}$

So the answer is no.

76. Let A_n be the event that E occured on the nth. trials and E,F didn't occur on the first $n-1$ trials

Let A be the event that E will occur before F.

$p(A_n) = p(E)\cdot(1-p(E)-p(F))^{n-1}$ and $p(A) = \sum_{n=1}^{\infty} p(A_n) = p(E)\sum_{n=1}^{\infty}(1-p(E)-p(F))^{n-1}$

$\therefore\ p(A) = p(E)\cdot\dfrac{1}{1-(1-p(E)-p(F))} = \dfrac{p(E)}{p(E)+p(F)}$ by using the formula $\sum_{n=1}^{\infty} a^n = \dfrac{a}{1-a}$ if $|a| < 1$

86. (a) Using the hint, let $N(B)$ denote the number of elements in B

so $p(A\subset B) = \sum_{i=0}^{n} p(A\subset B \mid N(B)=i)\, p(N(B)=i)$.

because $S = \{1, 2 \cdots n\}$ and A, B are equally likely to be any of the 2^n subsets of S

so B from binomial distribution $B(n, \frac{1}{2})$ so $P(N(B)=i) = \binom{n}{i}\dfrac{1}{2^n}$

$p(A\subset B \mid N(B)=i) = \dfrac{2^i}{2^n}$ $\quad$ so $p(A\subset B) = \sum_{i=0}^{n}\dfrac{2^i}{2^n}\cdot\binom{n}{i}\cdot\dfrac{1}{2^n} = \dfrac{1}{4^n}\sum_{i=0}^{n} 2^i 1^{n-i}\binom{n}{i}$

$= \dfrac{1}{4^n}(1+2)^n = \left(\dfrac{3}{4}\right)^n.$

(b) $p(AB=\phi) = p(A\subset B^c) = p(A\subset B) = \left(\dfrac{3}{4}\right)^n$ because B^c is also equally likely to be any of the subsets

4. Let N_i denote the event that the ball is not found in a search of box i, and Let B_j denote the event that it is in box j.

$$p(B_j|N_i) = \frac{p(N_i|B_j)p(B_j)}{p(N_i|B_j)p(B_j) + p(N_i|B_j^c)p(B_j^c)}$$

$p(B_j) = P_j$ and $p(N_i|B_j) = 1$ if $j \neq i$

and $p(N_i|B_j) = 1 - \alpha_i$ if $j = i$ $\quad p(N_i|B_i^c) = 1$

so $p(B_j|N_i) = \dfrac{P_j}{(1-\alpha_i)P_i + 1\cdot(1-P_i)} = \dfrac{P_j}{1-\alpha_i P_i}$ if $j \neq i$

$p(B_j|N_i) = \dfrac{(1-\alpha_i)P_i}{(1-\alpha_i)P_i + 1 - P_i} = \dfrac{(1-\alpha_i)P_i}{1-\alpha_i P_i}$ if $j = i$

6 $p(E_1 \cup E_2 \cup \cdots \cup E_n) = 1 - p((E_1 \cup E_2 \cdots \cup E_n)^c) = 1 - p(E_1^c \cap E_2^c \cap \cdots \cap E_n^c)$

because $E_1, E_2, \cdots, E_n$ are independent events so $p(E_1^c \cap E_2^c \cap \cdots \cap E_n^c) = p(E_1^c)p(E_2^c)\cdots p(E_n^c)$

$\therefore p(E_1 \cup E_2 \cup \cdots \cup E_n) = 1 - p(E_1^c)p(E_2^c)\cdots p(E_n^c) = 1 - \prod_{i=1}^{n}(1 - p(E_i))$

7 (a) They will be all white if the last ball withdrawn from the urn (when all balls are withdrawn) is white. so the answer will be the same as the probability that the last ball withdrawn from the urn is white. Since the probability of drawing a white ball on the ith ball selected is the same as the probability of drawing a white ball on the first ball selected, this probability is $\frac{n}{n+m}$.

9. $p(AB) = p(\text{both Land heads}) = \frac{1}{4} = \frac{1}{2}\cdot\frac{1}{2} = p(A)\cdot p(B)$.

$p(AC) = p(\text{both Land on the same side and first coin Lands head}) = p(\text{both land heads}) = \frac{1}{4}$

$= \frac{1}{2}\cdot\frac{1}{2} = p(A)\cdot p(C)$

$p(BC) = p(\text{both land heads}) = \frac{1}{4} = \frac{1}{2}\cdot\frac{1}{2} = p(B)p(C)$

But $p(ABC) = p(\text{both land heads}) = \frac{1}{4} \neq p(A)p(B)p(C) = \frac{1}{8}$

11. $p(\text{At least one head after } n \text{ tosses}) = 1 - p(\text{all tails after } n \text{ tosses}) = 1 - (1-p)^n \geq \frac{1}{2}$

so $(1-p)^n \leq \frac{1}{2}$ $\qquad n \geq -\dfrac{\log 2}{\log(1-p)}$

13. Condition on the initial flip. If it lands on heads then A will accumulates n heads before B accumulates m with probability $P_{n-1,m}$, whereas if it lands tails then B will start first and B accumulates m heads before A accumulates n heads with probability $P_{m,n}$. so A accumulates n heads before B accumulates m heads with probability $1 - P_{m,n}$.

$P_{n,m}$ = P(A accumulates n heads before B accumulates m heads)

= P(A first flips heads) · P(A accumulates n heads before B accumulates m heads | A first flips heads) + P(A first flips tails) P(A accumulates n heads before B accumulates m heads | A first flips tails)

$= p \cdot P_{n-1,m} + (1-p) \cdot P_{m,n}$.

15. In order for it to take n trials to obtain r successes, $r-1$ successes must occur in the first $n-1$ trials and the rth success occurs on the nth trial.

there are $\binom{n-1}{r-1}$ possibilities to have $r-1$ successes in the first $n-1$ trials

so P(exactly n trials are required) $= \binom{n-1}{r-1} p^r (1-p)^{n-r}$

16 Condition on the ~~initial~~ first $n-1$ trials, if the ~~initial~~ first $n-1$ trials result in an even number of successes, the probability that n trials result in an even number of successes is $1-p$, and the probability that the first $n-1$ trials result in an even number of successes is P_{n-1}, if the first $n-1$ trials result in an odd number of successes, the probability that n trials result in an even number of successes is p so P_n = P(even successes in n trials) =

P(even successes in n trials | even successes in $n-1$ trials) P(even successes in $n-1$ trials)
+ P(even successes in n trials | odd success in $n-1$ trials) P(odd successes in $n-1$ trials)

$= (1-p) P_{n-1} + p \cdot (1-P_{n-1}) = p + (1-2p) P_{n-1}$. $P_0 = 1$

~~Assum~~ We want to prove $P_n = \dfrac{1+(1-2p)^n}{2}$.

First step: prove when $n=1$, the probability that first trial result in even number of successes is $1-p$ because this even number must be 0 so the first trial is a failure.

And $1-p = P_1 = \dfrac{1+(1-2p)^1}{2} = 1-p$ so the formula is correct when $n=1$

Second step: We assume ~~the~~ $P_n = \dfrac{1+(1-2p)^n}{2}$ holds when $n=k$ trials.

so $P_k = \dfrac{1+(1-2p)^k}{2}$ using $P_{k+1} = p + (1-2p) P_k$.

When $n=k+1$ $P_{k+1} = p + (1-2p) \cdot \dfrac{1+(1-2p)^k}{2} = p + \dfrac{1-2p}{2} + \dfrac{(1-2p)^{k+1}}{2} = \dfrac{1+(1-2p)^{k+1}}{2}$

$\therefore$ $P_n = \dfrac{1+(1-2p)^n}{2}$ holds when $n=k+1$.

So the formula $P_n = \dfrac{1+(1-2p)^n}{2}$ holds for $n = 0, 1, 2 \cdots$

20. α_n denote the probability that the nth ball is chosen from the first urn.

$\alpha_{n+1} = p($ n+1th ball is chosen from the first urn $)$

$= p($ n+1 th ball is chosen from the first urn $|$ nth ball is chosen from the first urn $) \times$
$p($ nth ball is chosen from the first urn $) + p($ n+1 th ball is chosen from the first urn $|$
nth ball is chosen from the second urn $) \cdot p($ nth ball is chosen from the second urn $)$

$= p \cdot \alpha_n + (1-p')(1-\alpha_n) = \alpha_n(p+p'-1) + (1-p')$

because $p($ n+1 th ball is chosen from the first urn $|$ nth ball is chosen from the first urn $)$

$= p($ nth ball from the first urn is white $) = p.$

$p($ n+1 th ball is chosen from the first urn $|$ nth ball is chosen from the second urn $)$

$= p($ nth ball from the second urn is black $) = 1-p'$

$\alpha_1 = p($ 1th ball is chosen from the first urn $) = \alpha.$

By induction. when $n=1.$ $\alpha_1 = \frac{1-p'}{2-p-p'} + \alpha - \frac{1-p'}{2-p-p'} = \alpha$ the. ~~fo~~ equation:

$\alpha_n = \frac{1-p'}{2-p-p'} + \left(\alpha - \frac{1-p'}{2-p-p'}\right)(p+p'-1)^{n-1}$ holds

Assume. when $n=k$, $\alpha_k = \frac{1-p'}{2-p-p'} + \left(\alpha - \frac{1-p'}{2-p-p'}\right)(p+p'-1)^{k-1}$

so when $n=k+1$, $\alpha_{k+1} = \alpha_k(p+p'-1) + 1-p' = (p+p'-1)\frac{1-p'}{2-p-p'} + \left(\alpha - \frac{1-p'}{2-p-p'}\right)(p+p'-1)^k$
$+ 1-p' = (1-p')\left(\frac{p+p'-1}{2-p-p'} + 1\right) + \left(\alpha - \frac{1-p'}{2-p-p'}\right)(p+p'-1)^k = \frac{1-p'}{2-p-p'} + \left(\alpha - \frac{1-p'}{2-p-p'}\right)(p+p'-1)^k$

so $\alpha_n = \frac{1-p'}{2-p-p'} + \left(\alpha - \frac{1-p'}{2-p-p'}\right)(p+p'-1)^{n-1}$ $n=1,2\ldots$

Because $0 \leq p, p' \leq 1$ so $|p+p'-1| < 1$ so $(p+p'-1)^{n-1} \to 0$ as $n \to \infty$.

so $\lim_{n\to\infty} \alpha_n = \frac{1-p'}{2-p-p'}$

P_n denote nth ball is white.

$P_n = p($ n th ball is white $|$ ~~nth ball is white~~ nth ball from the first urn $)$ $p($ nth ball from the first urn $)$
$+ p($ n th ball is white $|$ nth ball from the second urn $)$ $p($ nth ball from the second urn $)$

$= p \cdot \alpha_n + \cancel{(1-p)}\ p'(1-\alpha_n)$

$= p\frac{1-p'}{2-p-p'} + p\left(\alpha - \frac{1-p'}{2-p-p'}\right)(p+p'-1)^{n-1} + p' - p'\frac{1-p'}{2-p-p'} - p'\left(\alpha - \frac{1-p'}{2-p-p'}\right)(p+p'-1)^{n-1}$

$= \frac{p+p'-2pp'}{2-p-p'} + (p-p')\left(\alpha - \frac{1-p'}{2-p-p'}\right)(p+p'-1)^{n-1}$

as $|p+p'-1|<1$ so $(p+p'-1)^{n-1} \to 0$ as $n\to\infty$

$\therefore \lim_{n\to\infty} P_n = \dfrac{p+p'-2pp'}{2-p-p'}$

22. $P_n = p$(it will be dry n days later | the weather is dry on Jan 1.).

$=p$(it will be dry n days later | the weather will be dry $n-1$ days later, dry on Jan 1)

$\cdot\, p$(it will be dry $n-1$ days later | dry on Jan 1) $+ p$(dry n days later | it will be wet $n-1$ days later, dry on Jan 1) p(it will be wet $n-1$ days later | dry Jan 1)

$= p\cdot P_{n-1} + (1-p)(1-P_{n-1}) = (2p-1)P_{n-1} + (1-p)$

By induction. $P_0 = 1$. when $n=1$ $P_1 = p$(it will be dry 1 days later | dry on Jan 1) $=$

$= \cancel{p\cdot P_0 + (1-p)(1-P_0) = p}\ \frac{1}{2} + \frac{1}{2}(2p-1)^1 = p$

the equation $P_n = \frac{1}{2} + \frac{1}{2}(2p-1)^n$ holds

Assume that when $n=k$ the equation holds so $P_k = \frac{1}{2} + \frac{1}{2}(2p-1)^k$

so $P_{k+1} = p\cdot P_k + (1-p)(1-P_k) = (2p-1)P_k + (1-p) = \frac{1}{2}(2p-1) + \frac{1}{2}(2p-1)^{k+1} + (1-p)$

$= \frac{1}{2} + \frac{1}{2}(2p-1)^{k+1}$ the equation holds

so $P_n = \frac{1}{2} + \frac{1}{2}(2p-1)^n$, $n = 1, 2 \cdots$

29. Let H denote the next m flips also result in all heads, F_n denote the first n flips all result in heads and C_i denote ith coin is selected.

so $p(H|F_n) = \sum_{i=0}^{k} p(H|F_n, C_i)\, p(C_i|F_n)$

$p(H|F_n, C_i) = p(H|C_i) = \left(\frac{i}{k}\right)^m$

$p(C_i|F_n) = \dfrac{p(F_n|C_i)\cdot p(C_i)}{\sum_{i=0}^{k} p(F_n|C_i)p(C_i)} = \dfrac{(i/k)^n \cdot \frac{1}{k+1}}{\sum_{i=0}^{k} (i/k)^n \cdot \frac{1}{k+1}} = \dfrac{i^n}{\sum_{i=0}^{k} i^n}$

so $p(H|F_n) = \sum_{i=0}^{k} \left(\frac{i}{k}\right)^m \cdot \dfrac{i^n}{\sum_{j=0}^{k} j^n} = \dfrac{\sum_{i=0}^{k} \left(\frac{i}{k}\right)^{n+m}}{\sum_{j=0}^{k} \left(\frac{j}{k}\right)^n} \approx \dfrac{\int_0^1 x^{n+m}\,dx}{\int_0^1 x^n\,dx} = \dfrac{n+1}{n+m+1}$