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3 If f is a probability density function, then $f(x) \geq 0 \quad \forall x \in R$

when $f(x) = \begin{cases} c(2x-x^3), & 0<x<\frac{5}{2} \\ 0, & \text{otherwise} \end{cases}$, $\because f(\frac{5}{2}) = c(5-\frac{1}{8}125) = -\frac{85}{8}\cdot c$ and $f(1) = c$

then $-\frac{25}{8}c \geq 0$ and $c \geq 0$ $\therefore c = 0$

then $f(x) = 0$, $x \in \mathbb{R}$, this is not reasonable. so $f(x) = \begin{cases} c(2x-x^3), & 0<x<\frac{5}{2} \\ 0, & \text{otherwise.} \end{cases}$

For $f(x) = \begin{cases} c(2x-x^2) & 0<x<\frac{5}{2} \\ 0 & \text{otherwise.} \end{cases}$ $f(\frac{5}{2}) = -\frac{5}{4}c \geq 0$, $f(1) = c \geq 0$ $\therefore c = 0$

But $f(x) = 0$ is not reasonable, so $f(x) = \begin{cases} c(2x-x^2), & 0<x< \\ 0, & \text{o.w.} \end{cases}$

is not a probability density.

11. If $x > \frac{L}{2}$, then the ratio of the shorter to the longer is $\frac{L-x}{x}$ and $\frac{L-x}{x} < 1$, $\frac{x}{L-x} > 1$

If $x \leq \frac{L}{2}$, then the ratio of the shorter to the longer is $\frac{x}{L-x}$ and $\frac{x}{L-x} < 1$

so the ratio of the shorter to the longer is $\min(\frac{L-x}{x}, \frac{x}{L-x})$.

the $p(\min(\frac{L-x}{x}, \frac{x}{L-x}) < 1/4) = 1 - p(\min(\frac{L-x}{x}, \frac{x}{L-x}) > 1/4) = 1 - p(\frac{L-x}{x} > \frac{1}{4}, \frac{x}{L-x} > \frac{1}{4})$

$= 1 - p(x > L/5, x < 4L/5) = 1 - p(\frac{L}{5} < x < 4L/5) = 1 - \int_{\frac{L}{5}}^{\frac{4}{5}L} \frac{1}{L} dx = 1 - \frac{3}{5} = \frac{2}{5}$.

13 Let X denote the time you will have to wait and $X \sim U(0, 30)$

a) so $P(X > 10) = \int_{10}^{30} \frac{1}{30} dx = \frac{2}{3}$

b) $p($ you will have to wait an additional 10 minutes $|$ at 10:15 the bus has not yet arrived$)$

$= p(X > 25 \mid X > 15) = \frac{P(X>25)}{P(X>15)} = \frac{\int_{25}^{30} \frac{1}{30} dx}{\int_{15}^{30} \frac{1}{30} dx} = \frac{\frac{5}{30}}{\frac{15}{30}} = \frac{1}{3}$

31. a) $E(|X-a|) = \int_0^A |x-a| \frac{1}{A} dx = \int_a^A (x-a)\frac{1}{A} dx + \int_0^a (a-x)\frac{1}{A} dx = \frac{1}{A} \cdot \frac{(x-a)^2}{2}\Big|_a^A + \frac{1}{A} \frac{-(x-a)^2}{2}\Big|_0$

$= \frac{(A-a)^2}{2} \cdot \frac{1}{A} + \frac{1}{A} \cdot \frac{a^2}{2}$

$\frac{d\, E(|X-a|)}{da} = \frac{1}{2A} 2(A-a)(-1) + \frac{1}{2A} \cdot 2a = 0 \implies a = \frac{A}{2}$ to minimize $E(|X-a|)$.

b) $E(|X-a|) = \int_0^{\cancel{A}\infty} |x-a| \lambda e^{-\lambda x} dx = \int_0^a (a-x)\lambda e^{-\lambda x} dx + \int_a^\infty (x-a)\lambda e^{-\lambda x} dx$.

$= a\lambda \int_0^a e^{-\lambda x} dx - \int_0^a \lambda x e^{-\lambda x} dx + \int_a^\infty \lambda x e^{-\lambda x} dx - a \int_a^\infty \lambda e^{-\lambda x} dx$

$= a\lambda \cdot \frac{e^{-\lambda x}}{-\lambda}\Big|_0^a - \lambda x \cdot \frac{e^{-\lambda x}}{-\lambda}\Big|_0^a + \int_0^a \frac{e^{-\lambda x}}{-\lambda} \cdot \lambda\, dx + \lambda x \cdot \frac{e^{-\lambda x}}{-\lambda}\Big|_a^\infty - \int_a^\infty \lambda \frac{e^{-\lambda x}}{-\lambda} dx - a\lambda \frac{e^{-\lambda x}}{-\lambda}$

$$= -ae^{-\lambda a} + a + ae^{-\lambda a} - a - \frac{e^{-\lambda x}}{-\lambda}\Big|_0^a + ae^{-\lambda a} + \frac{e^{-\lambda x}}{-\lambda}\Big|_a^\infty - ae^{-\lambda a}$$

$$= \frac{e^{-\lambda a}}{\lambda} - \frac{1}{\lambda} + \frac{e^{-\lambda a}}{\lambda} = \frac{2}{\lambda}e^{-\lambda a} - \frac{1}{\lambda} = 0 \quad \therefore e^{-\lambda a} = \frac{1}{2} \quad a = \frac{\log 2}{\lambda}$$ to minimize $E(|X-a|)$.

34 Let X denote the distance ~~Jones~~ the car can drive. (in thousands of miles)

a) $X \sim \exp(\frac{1}{20})$

P(She could get at least 20,000 miles | have been driven 10,000 miles) $= P(X>30 \mid X>10)$

$$= \frac{\frac{1}{20}e^{-\frac{1}{20}\cdot 30}}{\frac{1}{20}e^{-\frac{1}{20}\cdot 10}} = e^{-1}$$

b) $X \sim U(0,40)$

P(she could get at least 20,000 miles | have been driven 10,000 miles) $= P(X>30 \mid X>10)$

$$= \frac{P(X>30)}{P(X>10)} = \frac{\int_{30}^{40}\frac{1}{40}dx}{\int_{10}^{40}\frac{1}{40}dx} = 1/3$$

36 the lifttime X has exponential distribution with parameter $\lambda(t) = t^3$, $t>0$

then $F(x) = P(X \le x)$ ~~$= 1 - e^{-\lambda(t)x}$~~ $= 1 - \exp\{-\int_0^t \lambda(t)dt\}$

a) $P(X \ge 2) = 1 - P(X \le 2) = \exp\{-\int_0^2 t^3 dt\} = e^{-4}$

b) P(life time is between .4 and 1.4) $= F(1.4) - F(.4) = e^{-\int_0^{0.4}t^3dt} - e^{-\int_0^{1.4}t^3dt}$

$$= e^{[-(.4)^4/4]} - e^{[-(1.4)^4/4]}$$

c) $$P(X \ge 2 \mid X \ge 1) = \frac{P(X\ge 2)}{P(X \ge 1)} = \frac{e^{-\int_0^2 t^3dt}}{e^{-\int_0^1 t^3 dt}} = e^{-\frac{15}{4}}$$

38 For both roots to be real the discriminant $(4Y)^2 - 16(Y+2)$ must be ≥ 0

and $(4Y)^2 - 16(Y+2) = 16(Y^2-Y-2) \ge 0 \Rightarrow Y \ge 2$ and $Y \le -1$ and $Y \sim U(0,5)$.

so Y should greater than 0 $\therefore Y \ge 2$ so the probability is $P(Y \ge 2) = \int_2^5 \frac{1}{5}dy = \frac{3}{5}$

P251 ~ P254

2. $$\int_0^\infty P(Y<-y)dy = \int_0^\infty\int_{-\infty}^{-y} f_Y(x)\,dx\,dy = \int_{-\infty}^0\int_0^{-x} f_Y(x)\,dy\,dx = \int_{-\infty}^0 -x f_Y(x)dx.$$

$$\int_0^\infty P(Y>y)dy = \int_0^\infty\int_y^\infty f_Y(x)\,dx\,dy = \int_0^\infty\int_0^x f_Y(x)\,dx\,dy = \int_0^\infty x f_Y(x)dx$$

$$\therefore E(Y) = \int_{-\infty}^\infty x f_Y(x)dx = \int_0^\infty x f_Y(x)dx - \int_{-\infty}^0 -x f_Y(x)dx = \int_0^\infty P(Y>y)dy - \int_0^\infty P(Y<-y)dy$$

3. Use problem 2, $E[g(X)] = \int_0^\infty P(g(X) > y)\,dy - \int_0^\infty P(g(X) < -y)\,dy$

$\int_0^\infty P(g(X) \geq y)\,dy = \int_0^\infty \int_{\{x: g(x) \geq y\}} f(x)\,dx\,dy = \int_{\{x: g(x) \geq 0\}} \int_0^{g(x)} f(x)\,dy\,dx = \int_{\{x: g(x) \geq 0\}} f(x) g(x)\,dx$

$\int_0^\infty P(g(X) < -y)\,dy = \int_0^\infty \int_{\{x: g(x) < -y\}} f(x)\,dx\,dy = \int_{\{x: g(x) < 0\}} \int_0^{-g(x)} f(x)\,dy\,dx = -\int_{\{x: g(x) < 0\}} f(x) g(x)\,dx$

$\therefore E[g(X)] = \int_{\{x: g(x) \geq 0\}} f(x) g(x)\,dx + \int_{\{x: g(x) < 0\}} f(x) g(x)\,dx = \int_R f(x) g(x)\,dx.$

5. $E[X^n] = \int_0^\infty P(X^n > t)\,dt$ Let $t = x^n$. then $E[X^n] = \int_0^\infty P(X^n > x^n)\,dx^n = \int_0^\infty n x^{n-1} P(X^n > x^n)\,dx$

Now we need to do is to prove that $X^n > x^n \Longleftrightarrow X > x$. Since $X^n - x^n = (X - x) \cdot \sum_{i=1}^{n-1} X^{n-i} x^i$

and X, x both positive, so $\sum_{i=1}^{n} X^{n-i} x^i > 0$ so $X^n - x^n > 0 \Longleftrightarrow X - x > 0$

so $E[X^n] = \int_0^\infty n x^{n-1} P(X > x)\,dx.$

6 $E_a = (0,1) \setminus \{a\}$ interval $(0,1)$ exclude one point $\{a\}$, $0 < a < 1$ and X be uniform over $(0,1)$

so $P(E_a) = \int_0^a 1\,dx + \int_a^1 1\,dx = a + 1 - a = 1$ for all a

$P(\bigcap_a E_a) = 1 - P(\bigcup_a E_a^c) = 1 - P(\bigcup_a \{a\}) = 1 - P(0,1) = 1 - 1 = 0$

so the collection of events E_a have the property that $P(E_a) = 1$, $P(\bigcap_a E_a) = 0$

8 First $E[X^2] \leq C E[X]$ because $E[X^2] = \int_0^C x^2 f(x)\,dx \leq \int_0^C C x f(x)\,dx = C E[X]$ $(X \leq C)$

then $Var(X) = E[X^2] - [E[X]]^2 \leq C E[X] - [E[X]]^2$ if we Let $\alpha = \frac{E[X]}{C}$

then $Var(X) \leq C^2 \alpha - (\alpha C)^2 = C^2 \alpha (1 - \alpha)$

and $C^2 \alpha (1 - \alpha)$ have maximum value $\frac{C^2}{4}$ because. $\frac{d[C^2 \alpha (1-\alpha)]}{\alpha} = C^2 \cdot [1 - \alpha - \alpha] = 0$ $\therefore \alpha = \frac{1}{2}$

then $C^2 \alpha (1 - \alpha) = \frac{C^2}{4}$ so $Var(X) \leq C^2 \alpha (1 - \alpha) \leq \frac{C^2}{4}$

24 Suppose X is Weibull with parameters ν, α, β. Then $F(x) = \begin{cases} 0 & x \leq \nu \\ 1 - \exp\{-(\frac{x-\nu}{\alpha})^\beta\} & x > \nu \end{cases}$

then $P(Y \leq x) = P((\frac{X-\nu}{\alpha})^\beta \leq x) = P(\frac{X-\nu}{\alpha} \leq x^{\frac{1}{\beta}}) = P(X \leq \alpha x^{\frac{1}{\beta}} + \nu) = F(\alpha x^{\frac{1}{\beta}} + \nu)$

$= 1 - \exp\{-(\frac{\alpha x^{\frac{1}{\beta}} + \nu - \nu}{\alpha})^\beta\} = 1 - \exp\{-x\}.$

then Y is an exponential random variable with parameter $\lambda = 1$.

28 $p(Y \le x) = p(F(X) \le x) = p(X \le F^{-1}(x)) = F(F^{-1}(x)) = x.$

$f(Y) = \dfrac{d\,p(Y \le x)}{dx} = 1$ and because $0 \le F(x) \le 1$, so $0 \le Y \le 1$

this means that Y is uniformly distributed over $(0,1)$.

29 There are two cases : (i) $\alpha > 0$ (ii) $\alpha < 0$ if $\alpha = 0$, $Y = b$ with probability 1

(i) $\alpha > 0$ $p(Y \le x) = p(aX + b \le x) = p(X \le \frac{x-b}{a}) = F_X(\frac{x-b}{a})$

$f_Y(x) = F_X'(\frac{x-b}{a}) \cdot (\frac{x-b}{a})' = f_X(\frac{x-b}{a}) \cdot \frac{1}{a}$

(ii) $\alpha < 0$ $p(Y \le x) = p(aX + b \le x) = p(X \ge \frac{x-b}{a}) = 1 - F_X(\frac{x-b}{a})$

$f_Y(x) = -F_X'(\frac{x-b}{a}) \cdot (\frac{x-b}{a})' = -\frac{1}{a} f_X(\frac{x-b}{a})$