Symmetrization for sums of independent stochastic processes.

Lemma 2.36. Let $Z_1, Z_2, \ldots, Z_n$ be independent stochastic processes with mean zero and indexed by $\mathcal{F}$, a class of functions. Then:

$$E^* \Phi\left(\frac{1}{2} \left\| \sum_{i=1}^{n} \varepsilon_i^\circ Z_i^\circ \right\|_{\mathcal{F}}\right) \leq E^* \Phi\left(\left\| \sum_{i=1}^{n} Z_i \right\|_{\mathcal{F}}\right)$$

$$\leq E^* \Phi\left(2 \left\| \sum \varepsilon_i^\circ (Z_i^\circ - \mu_i^\circ) \right\|_{\mathcal{F}}\right),$$

for every non-decreasing convex $\Phi : \mathbb{R} \to \mathbb{R}$ and arbitrary functions $\mu_i^\circ : \mathcal{F} \to \mathbb{R}$. Here $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are i.i.d. Rademacher variables and $(Z_1, Z_2, \ldots, Z_n)$ and $(\varepsilon_1, \ldots, \varepsilon_n)$ are independent. For computing outer expectations it is understood ~~that~~ the underlying prob. space is a product space $\prod_{i=1}^{n} (\mathcal{X}_i, \mathcal{A}_i, P_i) \times (\mathcal{Z}, \mathcal{C}, Q)$ and Z_i° is a function of the i'th co-ordinate of $(x, z) = (x_1, x_2, \ldots, x_n, z)$

By assumption $E(Z_i f) = 0 \quad \forall f \in \mathcal{F}$.

The processes $\{Z_i\}$ need not possess any measurability properties except the measurability of all finite dimensional marginals.

Proof of Lemma:

First show: $E^* \underline{\Phi}\left(\left\| \sum_{i=1}^{n} Z_i \right\|_{\mathcal{F}}\right)$

$$\leq E^* \underline{\Phi}\left(2 \left\| \sum_{i=1}^{n} \varepsilon_i (Z_i - \mu_i) \right\|_{\mathcal{F}}\right)$$

Proceed in the same way as in Lemma 2.3.1

So let $W_1, W_2, .., W_n$ be an independent copy of $Z_1, Z_2, .., Z_n$ (suitably defined on the product space

$$\left(\prod_{i=1}^{n} (\mathcal{X}_i, \mathcal{A}_i, P_i) \times (\mathcal{Z}, \mathcal{C}, Q) \times \prod_{i=1}^{n} (\mathcal{X}_i, \mathcal{A}_i, P_i)\right)$$

and depending on the last n co-ordinates, exactly as $(Z_1, Z_2, .., Z_n)$ depend on the first n co-ordinates.

Now: $\| \sum_{i=1}^{n} z_i^\circ f \|_{\mathcal{F}} = \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} z_i^\circ f \right|$

$$= \sup_{f \in \mathcal{F}} \left| \sum (z_i^\circ f - \mu_i^\circ f) - E_W^* (w_i^\circ f - \mu_i^\circ f) \right|$$

$$\leq E_W^* \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} (z_i^\circ - \mu_i^\circ) f - \sum (w_i^\circ - \mu_i^\circ) f \right|$$

$\because$ ϕ is non-decreasing, it follows that:

$$\phi\left(\| \sum_{i=1}^{n} z_i \|_{\mathcal{F}} \right) \leq \phi\left(E_W^* \| \sum (z_i - \mu_i) - \sum (w_i - \mu_i) \|_{\mathcal{F}} \right)$$

$$\underset{\text{(Jensen)}}{\leq} E_W \Big(\phi \big(\underbrace{\| \sum (z_i - \mu_i) - \sum (w_i - \mu_i) \|_{\mathcal{F}}^{*W}}_{\equiv T.} \big) \Big)$$

where $*W$ denotes the minimal measurable majorant of the sup. wrt $w_1, w_2, \dots, w_n$, with z_i's still fixed. $\because$ ϕ is non-decreasing and continuous the $*$ can be moved outside the bracket (i.e $\phi(T^*) = \phi(T)^*$), giving

$$\phi\left(\| \sum_{i=1}^{n} z_i^\circ \|_{\mathcal{F}} \right) \leq E_W^* (\phi(T))$$

$$T \equiv \| \sum (z_i^\circ - \mu_i^\circ) - \sum (w_i - \mu_i) \|_{\mathcal{F}}$$

Taking expectations w.r.t Z,

$$E^*_Z\, \phi(\|\sum z_i\|_{\mathcal{F}})$$

$$\leq E^*_Z E^*_W \left(\phi\left(\|\sum (z_i-\mu_i) - \sum (w_i-\mu_i)\|_{\mathcal{F}}\right)\right)$$

$$\leq E^*_{Z,W} \left(\phi\left(\|\sum (z_i-\mu_i) - \sum (w_i-\mu_i)\|_{\mathcal{F}}\right)\right)$$

(by Fubini's theorem for outer expectations)

Now note ~~that~~ for any $\{e_1, e_2, .., e_n\} \in \{-1, 1\}^n$,

$$E^*_{Z,W} \left(\phi\left(\|\sum e_i(z_i-\mu_i) - \sum e_i(w_i-\mu_i)\|_{\mathcal{F}}\right)\right)$$

$$= E^*_{Z,W} \left(\phi\left(\|\sum (z_i-\mu_i) - \sum (w_i-\mu_i)\|_{\mathcal{F}}\right)\right)$$

($\because$ multiplication by e_i when $e_i = -1$, flips $(w_i-\mu_i)$ and $(z_i-\mu_i)$ and $\{(z_i-\mu_i)_{i=1}^n, (w_i-\mu_i)_{i=1}^n\}$ has the same distribution as the vector obtained by swapping $(z_i-\mu_i)$ and $(w_i-\mu_i)$ for any number of co-ordinates i.)

Taking expectations on the right side (therefore) w.r.t the Rademachers yields:

$$E^* \Phi\left(\left\|\sum_{i=1}^n z_i\right\|_{\mathcal{F}}\right)$$

$$\leq E_\varepsilon E^*_{w,z}\left(\phi\left(\left\|\sum \varepsilon_i(z_i-\mu_i) - \sum \varepsilon_i(w_i-\mu_i)\right\|_{\mathcal{F}}\right)\right)$$

$$\leq E_\varepsilon E^*_{w,z}\left(\phi\left(\left\|\sum \varepsilon_i(z_i-\mu_i)\right\|_{\mathcal{F}} + \left\|\sum \varepsilon_i(w_i-\mu_i)\right\|_{\mathcal{F}}\right)\right)$$

$(\because \phi$ is non-decreasing$)$

$$\leq E_\varepsilon E^*_{w,z}\left(\frac{1}{2}\phi\left(\left\|2\sum \varepsilon_i(z_i-\mu_i)\right\|_{\mathcal{F}}\right) + \frac{1}{2}\phi\left(\left\|2\sum \varepsilon_i(w_i-\mu_i)\right\|_{\mathcal{F}}\right)\right)$$

by convexity

$$= \frac{1}{2} E_\varepsilon E^*_z\left(\phi\left(\left\|2\sum \varepsilon_i(z_i-\mu_i)\right\|_{\mathcal{F}}\right)\right) + \frac{1}{2} E_\varepsilon E^*_w\left(\phi\left(\left\|2\sum \varepsilon_i(w_i-\mu_i)\right\|_{\mathcal{F}}\right)\right)$$

$$= E^*\left(\Phi\left(2\left\|\sum \varepsilon_i(z_i-\mu_i)\right\|_{\mathcal{F}}\right)\right)$$

To tackle the other half:

$$E^*\left(\phi\left(\frac{1}{2}\|\sum \varepsilon_i z_i\|_{\mathcal{F}}\right)\right) \le E^*\Phi\left(\|\sum_{i=1}^{n} z_i\|_{\mathcal{F}}\right)$$

For any $\{e_1, e_2, \ldots, e_n\} \in \{-1, 1\}^n$, note ~~that~~:

$$\phi\left(\frac{1}{2}\|\sum e_i Z_i\|_{\mathcal{F}}\right)$$

$$= \phi\left(\frac{1}{2}\|\sum e_i (Z_i(f) - EW_i(f))\|_{\mathcal{F}}\right)$$

(with W_i's as before)

$$\le \phi\left(\frac{1}{2} E_W^* \|\sum e_i (Z_i - W_i) f\|_{\mathcal{F}}\right)$$

(by monotonicity of ϕ)

$$\le E_W^*\left(\phi\left(\|\frac{1}{2}\sum e_i (Z_i - W_i) f\|_{\mathcal{F}}\right)\right)$$

(using convexity and Problem 1.28)

The left-hand side is an average of terms of the form:

$$E_Z^* \phi\left(\frac{1}{2}\|\sum e_i Z_i\|_{\mathcal{F}}\right)$$ where

$\{e_1, e_2, \ldots, e_n\} \in \{-1, 1\}^n$.

(each n tuple gets probability 2^{-n})

$$\therefore \; E^*_Z \, \phi\left(\frac{1}{2} \| \sum e_i z_i \|_{\mathcal{F}}\right)$$

$$\le E_Z^* \, E_W^* \left(\phi\left(\| \frac{1}{2} \sum e_i (z_i - w_i) \|_{\mathcal{F}} \right) \right)$$

$$\le E^*_{Z,W} \left(\phi\left(\| \frac{1}{2} \sum e_i (z_i - w_i) \|_{\mathcal{F}} \right) \right)$$

$$= E^*_{Z,W} \left(\phi\left(\| \frac{1}{2} \sum (z_i - w_i) \|_{\mathcal{F}} \right) \right)$$

It follows that:

$$E\left(\phi\left(\frac{1}{2} \| \sum \varepsilon_i z_i \|_{\mathcal{F}} \right) \right) = E_\varepsilon \, E_Z^* \left(\phi\left(\frac{1}{2} \| \sum \varepsilon_i z_i \|_{\mathcal{F}} \right) \right)$$

$$\le E^*_{Z,W} \left(\phi\left(\| \frac{1}{2} \sum (z_i - w_i) \|_{\mathcal{F}} \right) \right)$$

$$\le E^*_{Z,W} \left(\phi\left(\frac{1}{2} \| \sum z_i \|_{\mathcal{F}} + \frac{1}{2} \| \sum w_i \|_{\mathcal{F}} \right) \right)$$

$$\le E^*_{Z,W} \left(\frac{1}{2} \phi\left(\| \sum z_i \|_{\mathcal{F}} \right) + \frac{1}{2} \phi\left(\| \sum w_i \|_{\mathcal{F}} \right) \right)$$

$$= \frac{E^*_Z \left(\phi\left(\| \sum_{i=1}^n z_i \|_{\mathcal{F}} \right) \right) + E^*_W \left(\phi\left(\| \sum_{i=1}^n w_i \|_{\mathcal{F}} \right) \right)}{2}$$

$$= E^* \left(\phi\left(\| \sum_1^n z_i \|_{\mathcal{F}} \right) \right).$$

In particular taking $\mu_i^\circ \equiv 0$ works.

Furthermore, taking $Z_i = \frac{\delta_{X_i} - P}{n}$ where $X_1, X_2, \ldots, X_n$ are i.i.d random variables and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are independent of $(X_1, X_2, \ldots, X_n)$, and $\mu_i \equiv 0$ we get:

$$E^* \Phi\left(\frac{1}{2} \left\| \frac{1}{n} \sum \varepsilon_i (f(X_i) - Pf) \right\|_{\mathcal{F}}\right)$$

$$\leq E^* \Phi\left(\|(\mathbb{P}_n - P)f\|_{\mathcal{F}}\right)$$

$$\leq E^* \Phi\left(2 \left\| \frac{1}{n} \sum \varepsilon_i (f(X_i) - Pf) \right\|_{\mathcal{F}}\right).$$

Lemma 2.3.6 shall prove useful later on in proving several equivalent statements about weak convergence.

Statement of Lemma 2.3.7:

Symmetrization for Probabilities:

For arbitrary stochastic processes $Z_1, Z_2, \ldots, Z_n$ and arbitrary functions $\mu_1, \ldots \mu_n : \mathcal{F} \to \mathbb{R}$

$$\beta_n(x)\, P^*\left(\|\sum_1^n Z_i^o\|_{\mathcal{F}} > x\right) \leq 2P^*\left(4\|\sum \varepsilon_i(Z_i - \mu_i)\|_{\mathcal{F}} > x\right),$$

for every $x > 0$ and $\beta_n(x) \leq \inf_f P\left(|\sum_1^n Z_i^o f| < \frac{x}{2}\right)$

In particular the above holds for i.i.d mean 0 processes and:

$$\beta_n(x) = 1 - \left(\frac{4n}{x^2}\right) \sup_f \operatorname{Var}(Z_1 f).$$

In deriving the above, we obtain the following inequality that we shall take recourse to, in obtaining Lemma 2.3.9. With $Y_1, \ldots, Y_n$ being an independent copy of $Z_1, Z_2, \ldots, Z_n$ we have:

$$\beta_n(x)\, P^*\left(\|\sum_{i=1}^n Z_i\|_{\mathcal{F}} > x\right) \leq P_Z^* P_Y^*\left(\|\sum (Y_i - Z_i)\|_{\mathcal{F}} > x/2\right)$$

Lemma 2.3.9.

Let $Z_1, Z_2, \ldots$ be i.i.d stochastic processes such that $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i$ converges weakly in $\ell^\infty(\mathcal{F})$ to a tight Gaussian process. Then:

$$\lim_{x \to \infty} x^2 \sup_n P^*\left(\frac{1}{\sqrt{n}} \left\| \sum_{i=1}^{n} Z_i \right\|_{\mathcal{F}} > x \right) = 0 \quad (A)$$

In particular $\|Z_1\|_{\mathcal{F}}^*$ is weak L_2, i.e

$$x^2 P^*(\|Z_1\|_{\mathcal{F}} > x) \to 0 \text{ as } x \to 0.$$

Furthermore the above result implies that

$$\sup_{n \geq 1} E^* \left\| \frac{1}{\sqrt{n}} \sum_{1}^{n} Z_i \right\|_{\mathcal{F}}^{r} < \infty \quad \forall\ 0 < r < 2$$

and this in turn implies the uniform integrability of $\left\{ \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i \right\|^* \right\}_{n=1}^{\infty}$.

<u>Proof</u>: Before proving Lemma 2.3.9 let's look at the last two assertions. To show that $\sup_{n \geq 1} E^* \left\| \frac{1}{\sqrt{n}} \sum_{1}^{n} Z_i \right\|_{\mathcal{F}}^{r} < \infty$ $\forall\ 0 < r < 2$, it suffices to show that

if $\{X_n\}$ is a sequence of non-negative random variables satisfying:

$\lim_{x\to\infty} x^2 \sup_{n\geq 1} P(X_n \geq x) = 0$, then

$\{E X_n^r\}$ is bounded.

Consider: $\sup_{n\geq 1} E X_n^r$

$$= \sup_{n\geq 1} \int_0^\infty r x^{r-1} P(X_n \geq x)\, dx$$

$$\leq \int_0^\infty r x^{r-1} \sup_{n\geq 1} P(X_n \geq x)\, dx. \to (B)$$

Now $\exists$ M so large such ~~that~~ $\forall x \geq M$,

$$x^2 \sup_{n\geq 1} P(X_n \geq x) \leq 1$$

$$\Rightarrow \sup_{n\geq 1} P(X_n \geq x) \leq \frac{1}{x^2}$$

Write (B) as:

$$\int_0^M r x^{r-1} \sup_{n\geq 1} P(X_n \geq x) + \int_M^\infty r x^{r-1} \sup_{n\geq 1} P(X_n \geq x)\, dx$$

$$\leq M^r + \int_M^\infty r x^{r-1} \frac{1}{x^2}\, dx$$

$$\leq M^r + \int_M^\infty \frac{dx}{x^{3-r}} < \infty$$

$\therefore\ 0 \le r < 2, \quad 3 - r > 1.$

To show uniform integrability of

$$\left\{ \left\| \frac{1}{\sqrt{n}} \sum_1^n z_i \right\|_{\mathcal{F}}^* \right\} = \{ f_n \} \text{ say,}$$

note that by what we have shown,

$$\sup_n \int |f_n|^{1+\varepsilon} \, dP < \infty \quad \text{for some } \varepsilon > 0$$

$$\overset{\|}{K} \text{ (say)}.$$

Then:
$$\int_{|f_n| \ge \alpha} |f_n| \, d\mu \le \frac{1}{\alpha^\varepsilon} \int_{|f_n| \ge \alpha} |f_n|^{1+\varepsilon} \, d\mu$$

$$\le \frac{K}{\alpha^\varepsilon} \quad \forall n$$

showing ~~that~~: $\sup_{n \ge 1} \int_{|f_n| \ge \alpha} |f_n| \, d\mu \to 0$ as $\alpha \to \infty$.

Proof proper: Let $Y_1, Y_2, \ldots, Y_n$ be an independent copy of $Z_1, Z_2, \ldots, Z_n$ defined as before (as co-ordinate projections) on an extra factor of a product probability space.

Now: $\frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i \xrightarrow[\text{weakly}]{} G$ (Gaussian proc) in $\ell^\infty(\mathcal{F})$

$\therefore \frac{1}{\sqrt{n}} \sum_{i=1}^{n} z_i(f) \longrightarrow N(*, *)$

implying ~~that~~ $E z_i(f) = 0$ and $Var(z_i(f)) < \infty$.

(converse to CLT) for every f. Thus the z_i's are mean 0 stochastic processes, and the marginal distributions of G, say the distribution of $(Gf_1, \ldots, Gf_k)$ is multivariate normal with covariance matrix given by that of $(Z_1 f_1, \ldots, Z_1 f_k)$.

Furthermore, by the characterization of weak convergence of a sequence of processes to a tight Gaussian element (page 41), it follows ~~that~~ $\mathcal{F}$ is totally bounded for the variance semi-metric $\rho(f,g) = \sigma(Z_1(f) - Z_1(g))$

$$= E(Z_1(f) - Z_1(g))^2.$$

In particular $\sigma^2 = \sup_{f \in \mathcal{F}} Var\, Z_1(f)$ is finite

(using total boundedness)

Using the intermediate step discussed earlier, we get:

$$\beta_n(x)\, P\left(\frac{1}{\sqrt{n}} \left\| \sum z_i \right\|_{\mathcal{F}}^{*} > x\right) \le P\left(\frac{1}{\sqrt{n}} \left\| \sum (z_i - y_i) \right\|_{\mathcal{F}}^{*} > \frac{x}{2}\right)$$

Here $\beta_n(x) = 1 - \frac{4n}{x^2} \sup_f \mathrm{Var}\left(\frac{z_1}{\sqrt{n}}(f)\right)$

$$= 1 - \frac{4\cancel{n}}{x^2} \cdot \frac{1}{\cancel{n}} \alpha^2$$

$$= 1 - \frac{4\alpha^2}{x^2}.$$

(we use the fact ~~that~~ $\frac{z_1}{\sqrt{n}}, \ldots, \frac{z_n}{\sqrt{n}}$ are i.i.d mean 0 processes).

$$\therefore P\left(\frac{1}{\sqrt{n}} \left\| \sum z_i \right\|_{\mathcal{F}}^{*} > x\right) \le \frac{1}{1 - \frac{4\alpha^2}{x^2}} P\left(\frac{1}{\sqrt{n}} \left\| \sum (z_i - y_i) \right\|_{\mathcal{F}}^{*} > \frac{x}{2}\right)$$

It follows ~~that~~:

$$\lim_{x \to \infty} x^2 \sup_{n \ge 1} P\left(\frac{1}{\sqrt{n}} \left\| \sum z_i \right\|_{\mathcal{F}}^{*} > x\right) \to 0$$

if:

$$\lim_{x \to \infty} x^2 \sup_{n \geq 1} P\left(\frac{1}{\sqrt{n}} \| \Sigma (z_i - y_i) \|_{\mathcal{F}}^* > x\right)$$

$$\to 0$$

and it suffices therefore to prove the assertion for symmetric z_i's. ($\because z_i - y_i \stackrel{d}{=} y_i - z_i$)

Let $\varepsilon > 0$ be given. If G denotes the limiting Gaussian process, then $\|G\|_{\mathcal{F}}$ has moments of all orders by Propn. A.2.3.

Propn. A.2.3: Let G be a mean 0, separable Gaussian process such ~~that~~ $\|G\|$ is finite a.s. Then:

$$E(\beta \|G\|^2) < \infty \text{ iff } \beta \, 2\sigma^2(G) < 1.$$

Now $\sigma^2(G)$ is precisely $\alpha^2 < \infty$ in our situation and $\therefore \exists \beta$ such that $E(\beta \|G\|^2) < \infty \Rightarrow$ existence of all moments of $\|G\|_{\mathcal{F}}$

$\Rightarrow$ $\| G \|_{\mathcal{F}}$ is weak L_2 i.e

$x^2 \, P(\| G \|_{\mathcal{F}} > x) \to 0$ as $x \to \infty$.

(easy to prove using the fact ~~that~~ :

$\int x^2 \, P(\| G \|_{\mathcal{F}} > x) \, dx < \infty$

$\Rightarrow \sum n^2 \, P(\| G \|_{\mathcal{F}} > n+1) < \infty$

$\Rightarrow$ $n^2 \, P(\| G \|_{\mathcal{F}} > n+1) \to 0$

Also $n \, P(\| G \|_{\mathcal{F}} > n+1) \to 0$ $\because E(\| G \|_{\mathcal{F}}^2) < \infty$

$\Rightarrow$ $(n+1)^2 \, P(\| G \|_{\mathcal{F}} > n+1) \to 0$ as $n \to \infty$

$\Rightarrow$ $x^2 \, P(\| G \|_{\mathcal{F}} > x) \to 0$ as $x \to \infty$.)

Thus $\exists \, x_0$ such ~~that~~ $\forall \, x \geq x_0$,

$P(\| G \|_{\mathcal{F}} \geq x) \leq \dfrac{\varepsilon}{x^2} < \dfrac{1}{8}$.

$\because \left\| \dfrac{1}{\sqrt{n}} \sum_1^n z_i \right\|_{\mathcal{F}} \xrightarrow[\text{weakly}]{} \| G \|_{\mathcal{F}}$,

by the Portmanteau theorem, it follows ~~that~~ :

$$\overline{\lim} P^* \left(\left\| \frac{1}{\sqrt{n}} \sum z_i \right\|_{\mathcal{F}} \geq x \right)$$

$$\leq P \left(\| G \|_{\mathcal{F}} \geq x \right)$$

$$\leq \frac{\varepsilon}{x^2} < \frac{1}{8}$$

$\Rightarrow \forall\, n$ ($\geq N$, for some N depending on ε and G).

$$P^* \left(\left\| \frac{1}{\sqrt{n}} \sum z_i \right\|_{\mathcal{F}} \geq x \right) \leq \frac{2\varepsilon}{x^2} < \frac{1}{4}.$$

Using Levy's inequalities A.1.2:

Propn: Let $X_1, X_2, \ldots, X_n$ be independent symmetric stochastic processes indexed by an arbitrary set. Then for every $\lambda > 0$

$$P^* \left(\max_{k \leq n} \| X_k \| > \lambda \right) \leq 2 P^* \left(\| S_n \| > \lambda \right)$$

Our z_i's are symmetric; hence so are our $\frac{z_i}{\sqrt{n}}$'s. Letting $\lambda = x$, $X_i = \frac{z_i}{\sqrt{n}}$, and invoking the above inequality we obtain

$$P\left(\max_{1\le i\le n} \|z_i\|_{\mathcal{F}}^* > x\sqrt{n}\right)$$

$$\le 2P\left(\left\|\sum_{i=1}^{n} z_i\right\|_{\mathcal{F}}^* > x\sqrt{n}\right) \le \frac{4\varepsilon}{x^2} < \frac{1}{2}$$

$$\forall\, n \ge N.$$

We shall now use the above inequality to deduce the behaviour of $\|z_1\|_{\mathcal{F}}^*$. Note ~~that~~ all the $\|z_i\|_{\mathcal{F}}^*$ are independent and identically distributed. Use the inequality:

$$\frac{\sum P(|\xi_i| > x)}{1 + \sum P(|\xi_i| > x)} \le P(\max|\xi_i| > x)$$

for independent r.v's $\xi_1, \xi_2, \ldots \xi_n$ to deduce ~~that~~,

1 if $P(\max|\xi_i| > x) < 1/2$,

then $\sum P(|\xi_i| > x) \le P(\max|\xi_i| > x)$

$$+ \frac{1}{2}\sum P(|\xi_i| > x)$$

i.e $\sum P(|\xi_i| > x) \le 2P(\max|\xi_i| > x)$.

Let $\xi_i = \|z_i\|_{\mathcal{F}}^*$. Then, we get:

$$n \, P\left(\|Z_1\|_{\mathcal{F}}^* > x\sqrt{n}\right)$$

$$\leq 2 \cdot P\left(\max_{1 \leq i \leq n} \|Z_i\|_{\mathcal{F}}^* > x\sqrt{n}\right)$$

$$< \frac{8\varepsilon}{x^2} \quad \forall\, n \,(\geq N)$$

$\therefore (\sqrt{n}\,x)^2 P\left(\|Z_1\|_{\mathcal{F}}^* > x\sqrt{n}\right) < 8\varepsilon \;\; \forall$ sufficiently large x $(\geq x_0)$, and $\forall\, n$.

$\Rightarrow t^2 P\left(\|Z_1\|_{\mathcal{F}}^* > t\right) \to 0$ as $t \to \infty$.

Thus we have shown that $\|Z_1\|_{\mathcal{F}}^*$ is weak L_2. To prove our original claim, we proceed as follows. Consider the following array of processes:

$$\begin{array}{cccccc}
Z_{11} & Z_{12} & Z_{13} & Z_{14} & \cdots \\
Z_{21} & Z_{22} & \cdots & & \\
Z_{23} & \vdots & \cdots & & \\
Z_{24} & \vdots & \cdots & &
\end{array}$$

where Z_{ij}'s are i.i.d as Z_1.

For each m, define:

$$\tilde{Z}_i^\circ(m) = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} Z_{ij}^{\circ\circ}$$

, the $\tilde{Z}_i^{(m)}$'s are independent and identically distributed and symmetric

Then, for each m,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{Z}_i^{(m)} = \frac{1}{\sqrt{nm}} \sum_{i=1}^{n} \sum_{j=1}^{m} Z_{ij}^{\circ\circ} \xrightarrow{d} G_r$$

as $n \to \infty$. $\Big\Vert\, d$

$$S_{nm} = \frac{1}{\sqrt{nm}} \sum_{i=1}^{mn} Z_i^\circ$$

Given $\varepsilon > 0$ as before, we can find x_0 and N (not depending on m) such that:

$$p^* \left(\Vert \sum \tilde{Z}_i^{\circ(m)} \Vert_{\mathcal{F}} > x\sqrt{n} \right) \le \frac{2\varepsilon}{x^2}, \quad n \geq N, \quad \forall\, x \geq x_0$$

(the N obtained before actually works) for every m.

Following the same chain of arguments as before, one obtains, that for every m:

$$(\sqrt{n}\,x)^2 \, P\left(\|\tilde{Z}_1^{(m)}\|_{\mathcal{F}}^* > x\sqrt{n}\right) < 8\varepsilon$$

$$\forall\, x \geq x_0,\ n \geq N$$

i.e $$(\sqrt{n}x)^2 \, P\left(\left\|\frac{1}{\sqrt{m}}\sum_{j=1}^{m} z_{ij}'\right\|_{\mathcal{F}}^* > x\sqrt{n}\right) < 8\varepsilon$$

i.e $$(\sqrt{n}x)^2 \sup_{m \geq 1} P\left(\left\|\frac{1}{\sqrt{m}}\sum_{i=1}^{m} z_i\right\|_{\mathcal{F}}^* > x\sqrt{n}\right) < 8\varepsilon$$

Thus: $$t^2 \sup_{m\geq 1} P\left(\left\|\frac{1}{\sqrt{m}}\sum_{i=1}^{m} z_i\right\|_{\mathcal{F}}^* > t\right) \to 0$$

as $t \to \infty$.

This finishes the proof of the Lemma

Lemma 2.3.11 : Let $Z_1, Z_2, \ldots$ be i.i.d mean 0,
^ Stochastic processes, linear in f. Set

$\rho_Z(f,g) = \sigma(Z_1(f) - Z_1(g))$ and $\mathcal{F}_\delta = \{f-g : \rho_Z(f,g) < \delta\}$

$= \sigma(Z_1(f-g))$

(tacit assumption regarding existence of second moments)

Then TFAE :

(i) $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i \Rightarrow$ a tight-limit in $\ell^\infty(\mathcal{F})$

(with Gaussian finite-dimensionals)

(ii) $(\mathcal{F}, \rho_Z)$ is totally bounded and

$\| \frac{1}{\sqrt{n}} \sum Z_i \|_{\mathcal{F}_{\delta_n}} \xrightarrow{P^*} 0$ for every $\delta_n \downarrow 0$

(ii') $(\mathcal{F}, \rho_Z)$ is totally bounded and for any $\varepsilon > 0$,

$$\lim_{\delta \downarrow 0} \overline{\lim_n} \, P^*\left(\| \frac{1}{\sqrt{n}} \sum Z_i \|_{\mathcal{F}_{\delta n}} > \varepsilon \right) = 0$$

(asymptotic equicontinuity condition)

(iii) $(\mathcal{F}, \rho_Z)$ is totally bounded and

$E^* \| \frac{1}{\sqrt{n}} \sum Z_i \|_{\mathcal{F}_{\delta_n}} \to 0$ for every $\delta_n \downarrow 0$

(iii) $(\mathcal{F}, \rho_Z)$ is totally bounded and

$$\lim_{\delta \downarrow 0} \overline{\lim_{n}} \, E^*\left(\left\| \frac{1}{\sqrt{n}} \sum_{1}^{n} Z_i \right\|_{\mathcal{F}_\delta}\right) = 0$$

for any given $\varepsilon > 0$

(iv') $(\mathcal{F}, \rho_Z)$ is totally bounded and $\forall \varepsilon > 0$

$$\lim_{\delta \downarrow 0} \overline{\lim_{n}} \, P^*\left(\left\| \frac{1}{\sqrt{n}} \sum \varepsilon_i Z_i \right\|_{\mathcal{F}_\delta} > \varepsilon\right) = 0 \quad (1)$$

where $\varepsilon_1, \varepsilon_2, \ldots$ are i i d Rademacher and independent of the Z_i's.

(iv) Same as above with (1) replaced by:

$$\left\| \frac{1}{\sqrt{n}} \sum \varepsilon_i Z_i \right\|_{\mathcal{F}_{\delta_n}} \xrightarrow{P^*} 0 \text{ for } \delta_n \downarrow 0$$

(v) $$E^* \left\| \frac{1}{\sqrt{n}} \sum \varepsilon_i Z_i \right\|_{\mathcal{F}_{\delta_n}} \to 0, \quad \delta_n \downarrow 0 \quad (2)$$

and $(\mathcal{F}, \rho_Z)$ is totally bounded (with ε_i's as above)

(v') Same as (v) with (2) replaced by

$$\lim_{\delta \downarrow 0} \overline{\lim_{n}} \, E^*\left(\left\| \frac{1}{\sqrt{n}} \sum \varepsilon_i Z_i \right\|\right) = 0$$

<u>Proof</u>: (i) and (ii') are equivalent using the necessary and sufficient conditions for convergence to a tight Gaussian process.

(ii') $\Leftrightarrow$ (ii), (iii) $\Leftrightarrow$ (iii'), (iv) $\Leftrightarrow$ (iv')

(v) $\Leftrightarrow$ (v') by standard delta-epsilon arguments (no probabilistic exercise involved)

(v) $\Rightarrow$ (iv), (iii) $\Rightarrow$ (ii) by Markov.

Since:

$$P^*\left(\left\| \frac{1}{\sqrt{n}} \sum \varepsilon_i z_i \right\|_{\mathcal{F}_{\delta_n}} > \varepsilon\right) \qquad (v) \Rightarrow (iv)$$

$$= P\left(\left\| \frac{1}{\sqrt{n}} \sum \varepsilon_i z_i \right\|^*_{\mathcal{F}_{\delta_n}} > \varepsilon\right)$$

$$\leq \frac{E^* \left\| \frac{1}{\sqrt{n}} \sum \varepsilon_i z_i \right\|_{\mathcal{F}_{\delta_n}}}{\varepsilon} \to 0$$

(iii) $\Leftrightarrow$ (iv) by dint of the fact that z_i's are independent mean 0 stochastic processes and we can invoke Lemma 2.3.6. to get:

$$\frac{1}{2} E^* \left\| \frac{\sum \varepsilon_i z_i}{\sqrt{n}} \right\|_{\mathcal{F}_{\delta_n}} \leq E^* \left\| \frac{1}{\sqrt{n}} \sum_1^n z_i \right\|_{\mathcal{F}_{\delta_n}}$$

$$\leq 2 E^* \left\| \frac{1}{\sqrt{n}} \sum \varepsilon_i z_i \right\|_{\mathcal{F}_{\delta_n}}$$

and now clearly (iii) $\Rightarrow$ (v) and (v) $\Rightarrow$ (iii)

Non trivial: (iii) $\Leftarrow$ (ii) and (iv) $\Rightarrow$ (v).

To show (iv) $\Rightarrow$ (v).

Note ~~that~~ if $\alpha^2 = \sup_f \mathrm{Var}\, Z_1(f)$, then:

by Lemma 2.3.7,

$$\left(1 - \frac{4\alpha^2}{x^2}\right) P^*\left(\left\| \frac{1}{\sqrt{n}} \sum Z_i \right\|_{\mathcal{F}_{\delta n}} > x \right)$$

$$\leq 2 P^*\left(\left\| \frac{1}{\sqrt{n}} \sum \varepsilon_i Z_i \right\|_{\mathcal{F}_{\delta n}} > \frac{x}{4} \right)$$

for every $x > 0$.

If (iv) is true then the R.H.S goes to 0

with n, showing ~~that~~:

$$P^*\left(\left\| \frac{1}{\sqrt{n}} \sum Z_i \right\|_{\mathcal{F}_{\delta n}} > x \right) \to 0 \quad ; \text{ so (ii)}$$

holds. But if (ii) $\Rightarrow$ (iii), we have

(iv) $\Rightarrow$ (v) $\because$ (iii) $\Rightarrow$ (v).

Remains to show ~~that~~ (ii) $\Rightarrow$ (iii).

We know that (ii) and (i) are equivalent.

Then by Lemma 2.3.9,

$$P^*(\|Z_1\|_{\mathcal{F}} > x) = o(x^{-2}) \text{ as } x \to \infty.$$

This implies that:

$$E^* \max_{1 \le i \le n} \frac{\|Z_i\|_{\mathcal{F}}}{\sqrt{n}} \to 0 \text{ by Problem 2.3.3.}$$

Prob 2.3.3. (i) If $\sup_{x > \varepsilon\sqrt{n}} n^{-1} \sum_{i=1}^{n} P(|X_{ni}| > x)x^2$

$\to 0$ for every $\varepsilon > 0$, then:

$$E \max_{1 \le i \le n} \frac{|X_{ni}|}{\sqrt{n}} \longrightarrow 0, \text{ where:}$$

$X_{n,1}, X_{n,2}, \ldots X_{n,n}$ is an arbitrary array of random variables.

If $(X_{n,1}, X_{n,2}, \ldots, X_{n,n})$

$= (X_1, X_2, \ldots, X_n)$ and the

X_i's are i.i.d then the above reduces to:

$\sup_{x > \varepsilon\sqrt{n}} x^2 P(|X_1| > x) \to 0$ for every $\varepsilon > 0$

$$\Rightarrow E \max_{1 \le i \le n} \frac{|X_i|}{\sqrt{n}} \to 0.$$

But $\sup_{x > \varepsilon\sqrt{n}} x^2 P(|X_1| > x) \to 0$ for every $\varepsilon > 0$

means ~~that~~ $x^2 P(|X_1| > x) \to 0$. and

vice-versa.

Here, take $X_i = \| Z_i \|_{\mathcal{F}}^*$ to get the result.

By the triangle inequality:

$$E^* \max_{1 \le i \le n} \frac{\| Z_i \|_{\mathcal{F}_{\delta n}}}{\sqrt{n}} \to 0 \quad \text{too.}$$

Now, apply Hoffman Jorgensen's inequality

as follows:

Let $p \geq 1$. and let $X_1, X_2, \ldots, X_n$ be

independent mean 0 stoch. processes.

Then $\exists$ constants K_p and $0 < v_p < 1$,

such that:

$$E^* \| S_n \|^p \le K_p \left(E^* \max_{i \le n} \| X_i \|^p + G^{-1}(v_p)^p \right)$$

G^{-1} being the quantile function of $\| S_n \|^*$.

Take $x_i = \frac{z_i}{\sqrt{n}}$, $p = 1$.

Then: $E^* \| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} z_i \|_{\mathcal{F}_{\delta n}}$

$$\leq K_1 \left(E^* \max_{1 \leq i \leq n} \frac{\| z_i \|_{\mathcal{F}_{\delta n}}}{\sqrt{n}} + G_n^{-1}(v_1) \right)$$

G_n is the quantile function of $\| \frac{1}{\sqrt{n}} \sum z_i \|^*_{\mathcal{F}_{\delta n}}$

Now $E^* \max_{1 \leq i \leq n} \frac{\| z_i \|_{\mathcal{F}_{\delta n}}}{\sqrt{n}} \to 0$ (shown).

$\because \| \frac{1}{\sqrt{n}} \sum_{1}^{n} z_i \|^*_{\mathcal{F}_{\delta n}} \xrightarrow{P} 0$ by (ii)

G_n^{-1}, the quantile function goes to 0

pointwise too.

This shows ~~that~~: $E^* \| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} z_i \|_{\mathcal{F}_{\delta n}} \to 0$

as $n \to \infty$.

$G_n^{-1}(s) = \inf \{ u : G_n(u) \geq s \}$