Change-Point Models Under Diminishing Mis-specification

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Introduction
Formulating the problem
Asymptotics

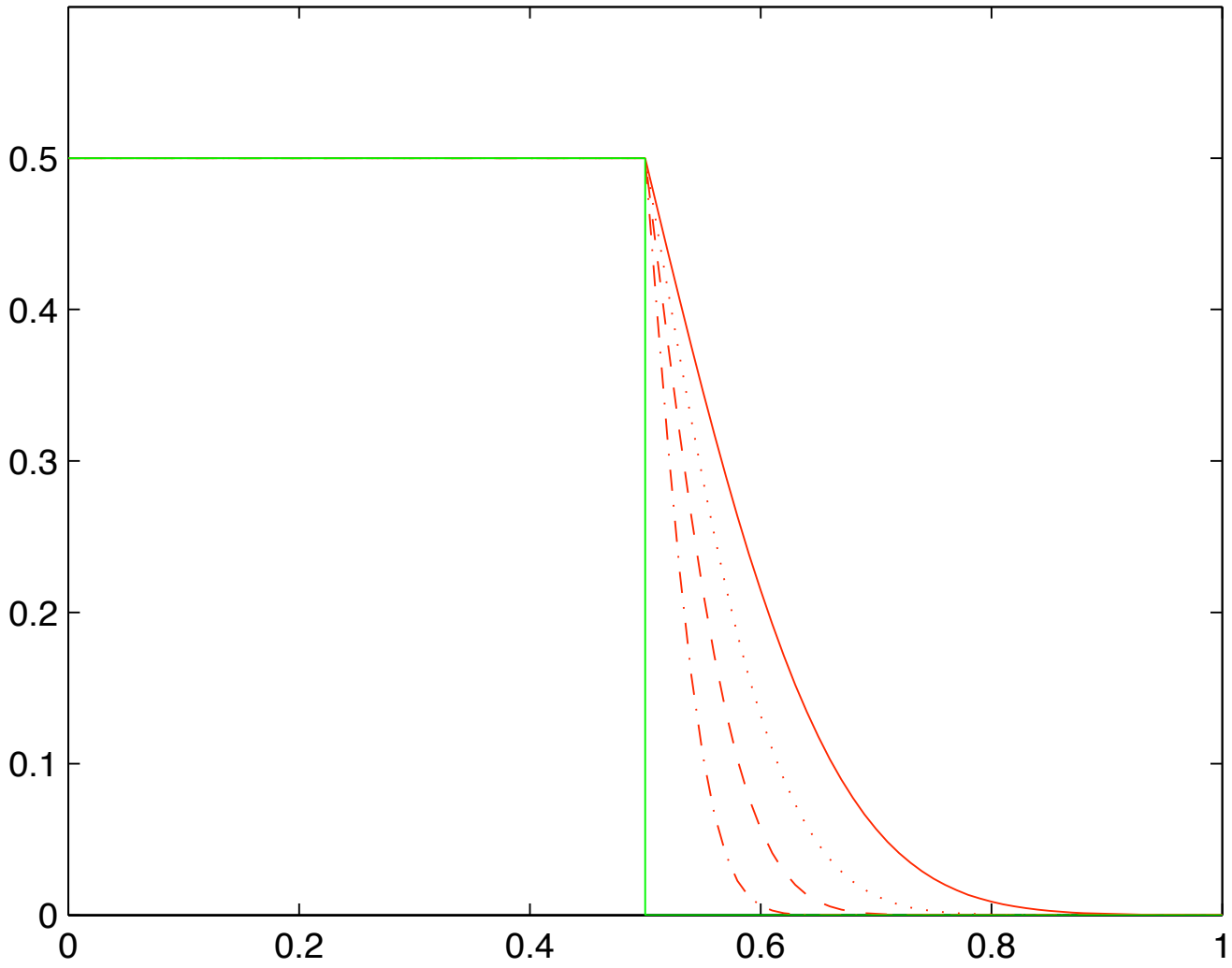
Mis-specified Change Point Model

- Data seem to come from a change-point model when in fact NOT.
- Can happen, for example, if the true regression function rises steeply over a small domain and is relatively flat otherwise and noise is not too large.
- Practitioner would still fit a change-point model.
- How do estimates of the change-point behave under such mis-specification?
- What are intelligent ways of modeling such mis-specification?

Fixed and Local Alternatives

- Think of a postulated change-point model, which for us will be a stump function, as the NULL. Thus, the null model is $\alpha\, 1(x \leq d) + \beta\, 1(x > d)$.
- Mis-specification can then be addressed in terms of either a fixed alternative or local alternatives.
- In the fixed alternative formulation, $Y = f(X) + \epsilon$ where f is a continuous (smooth) function.
- If the true (continuous) f is *close* to a stump function, modeling in terms of local alternatives is attractive (cue from classical parametric theory). In this formulation $Y = f_n(X) + \epsilon$ where f_n is a sequence of functions converging (appropriately) to the NULL.

Sequence of continuous functions converging to a stump



Questions

- What bearing does the speed of convergence of the local alternatives have on the (least-squares) estimates of the change-point?
- What can we say about (a) rates of convergence, and, (b) asymptotic distributions?
- Can we formulate adaptive procedures to estimate the change-point, i.e. procedures that do no necessitate *knowing the degree of mis-specification* in advance?
- Some related work due to Muller et. al., Huskova et.al. on gradually varying change-point models. However, in these settings, the NULL model postulates no change-point. The local alternatives exhibit diminishing jumps.
- Our point of view is the anti-thesis of the above. The situation is clearly not symmetric and different limit distributions obtain.

Fixed alternative

- Fit NULL model $f_{\alpha,\beta,d}(x) \equiv \alpha\, 1(x \leq d) + \beta\, 1(x > d)$ when true model is a fixed alternative $Y = f(X) + \epsilon$, f smooth.

$$(\hat{\alpha}, \hat{\beta}, \hat{d}) = \arg\min \mathbb{P}_n \left[Y - f_{\alpha,\beta,d}(X)\right]^2$$

consistently estimates

$$(\tilde{\alpha}, \tilde{\beta}, \tilde{d}) == \arg\min P \left[Y - f_{\alpha,\beta,d}(X)\right]^2 .$$

- Banerjee and McKeague (2007) show that $n^{1/3}(\hat{d} - \tilde{d})$ converges to a multiple of Chernoff's distribution. Problem also partly addressed by Buhlmann and Yu (2002).
- $\mathbb{Z} \equiv \arg\min_t \; W(t) + t^2$.
- In sharp contrast when the NULL model is true and the true parameters are (α_0, β_0, d^0), $n(\hat{d} - d^0)$ converges to a minimizer of a compound Poisson process. (Kosorok (2008), Ko and Qian (2009))

Formulation

- $Y = f_n(X) + \epsilon$ with $X \sim p_X$ on $[0, 1]$.

$$f_n(x) = f(\alpha_n(x - \theta_0))$$

  with $f : \mathbb{R} \mapsto [0, 1]$, continuously differentiable and $f(-\infty) = 0 = f(\infty) - 1$ and $0 < \theta_0 < 1$.
- Our NULL working model: $f_\theta(x) = \alpha\, 1(x \leq \theta) + \beta\, 1(x > \theta)$.
- α and β are known fixed quantitites with $0 \leq \alpha < \beta \leq 1$.
- We restrict attention to fitting 1 parameter models. The 3 parameter model will be discussed briefly later.

Formulation

- The least squares estimate:

$$\hat{\theta}^n = \arg\min_{\theta} \mathbb{P}_n \left[(Y-\alpha)^2 \, 1(X \leq \theta) + (Y-\beta)^2 \, 1(X > \theta) \right].$$

- Let

$$\mathbb{M}_n(\theta) = \mathbb{P}_n \left[(Y - (\alpha+\beta)/2) \, 1(X \leq \theta) \right].$$

- $\hat{\theta}^n = \arg\min_\theta \mathbb{M}_n(\theta)$ can be viewed as an estimator of $\theta^n = \arg\min_\theta M_n(\theta)$ where

$$M_n(\theta) = P_n \left[(Y - (\alpha+\beta)/2) \, 1(X \leq \theta) \right].$$

- $\theta^n = \theta_0 + \alpha_n^{-1} f^{-1}((\alpha+\beta)/2).$

Rates of convergence

- To determine (a non-trivial) r_n^α such that $r_n^\alpha\,(\hat{\theta}_n - \theta^n)$ is $O_p(1)$.
- Such a rate is determined via the interplay of two different quantities, these being:
  (a) the behavior of $M_n(\theta) - M_n(\theta^n)$ in an *appropriate* neighborhood of θ^n and
  (b) the behavior of the *expected modulus of continuity* of the process $\mathbb{G}_n\, f_n(x, y, \theta)$ in a neighborhood of θ^n.
- Here $\mathbb{G}_n = \sqrt{n}\,(\mathbb{P}_n - P_n)$ and

$$f_n(x, y, \theta) = (y - (\alpha + \beta)/2)\,(1(x \le \theta) - 1(x \le \theta^n))\,.$$

Rates of convergence

- Choose and fix $K > 0$. Under a mild assumption on f,

$$M_n(\theta) - M_n(\theta^n) \geq E_K \, \tilde{d}_n^2(\theta, \theta^n) \,,$$

for all $\tilde{d}_n(\theta, \theta^n) < \delta_0$, where

$$\tilde{d}_n^2(\theta, \theta^n) = \alpha_n \, (\theta - \theta_n)^2 \, 1(|\theta - \theta^n| \leq K/\alpha_n) + |\theta - \theta^n| \, 1(|\theta - \theta^n| > K/\alpha_n) \,.$$

- Via maximal inequalities, for $\delta \leq \delta_0$,

$$E^\star[\sup_{\tilde{d}_n(\theta,\theta^n)<\delta} |\mathbb{G}_n \, (f(x, y, \theta) - f(x, y, \theta^n))| \,] \leq \phi_n(\delta)$$

with

$$\phi_n(\delta) = \frac{\delta^{1/2}}{\alpha_n^{1/4}} \, 1 \left(\delta \leq \frac{1}{\sqrt{\alpha_n}} \right) + \delta \, 1 \left(\delta > \frac{1}{\sqrt{\alpha_n}} \right) \,.$$

Rates of Convergence

- Solving $r_n^2\,\phi_n(1/r_n) \le \sqrt{n}$ yields

$$\frac{r_n^{3/2}}{\alpha_n^{1/4}}\,1(r_n \ge \sqrt{\alpha_n}) + r_n\,1(r_n < \sqrt{\alpha_n}) \le \sqrt{n}\,. \qquad (1)$$

- **Case 1:** $\alpha_n = o(n)$. In this case, we seek a solution with $r_n \ge \sqrt{\alpha_n}$, whence the first term is relevant. A simple calculation shows $r_n = n^{1/3}\,\alpha_n^{1/6}$. Conclude that:

$$n^{1/3}\alpha_n^{1/6}(\alpha_n^{1/2}|\hat{\theta}_n - \theta^n|\,1(\alpha_n|\hat{\theta}_n - \theta^n| \le 1) + |\hat{\theta}_n - \theta^n|^{1/2}\,1(\alpha_n|\hat{\theta}_n - \theta^n| > 1))$$

  is $O_p(1)$. This implies that:

$$n^{1/3}\alpha_n^{2/3}|\hat{\theta}_n - \theta^n|\,1(\alpha_n|\hat{\theta}_n - \theta^n| \le 1) + n^{2/3}\,\alpha_n^{1/3}|\hat{\theta}_n - \theta^n|1(\alpha_n|\hat{\theta}_n - \theta^n| > 1)$$

  is $O_p(1)$.

- Since $n^{1/3}\alpha_n^{2/3}$ is slower than $n^{2/3}\alpha_n^{1/3}$, conclude that $n^{1/3}\alpha_n^{2/3}\,|\hat{\theta}_n - \theta^n| = O_p(1)$.

Rates of convergence

- **Case 2:** For $\alpha_n = n$ both rates $n^{1/3}\,\alpha_n^{2/3}$ and $n^{2/3}\,\alpha_n^{1/3}$ are equal to n and we conclude $n\,|\hat{\theta}_n - \theta^n|$ is $O_p(1)$.

- **Case 3:** For $n = o(\alpha_n)$, the second part in (1) becomes relevant i.e. we seek a solution with $r_n < \sqrt{\alpha}_n$. Conclude that:

$$(n\,\alpha_n)^{1/2}|\hat{\theta}_n - \theta^n|\,1(\alpha_n|\hat{\theta}_n - \theta^n| \leq 1) + n\,|\hat{\theta}_n - \theta^n| 1(\alpha_n|\hat{\theta}_n - \theta^n| > 1)\,.$$

Since $n\,\alpha_n$ is faster than n^2, it follows that $n\,|\hat{\theta}_n - \theta^n|$ is $O_p(1)$. On the other hand, by the observation that the least squares estimate $\hat{\theta}_n$ is atleast as far as θ^n from the X_i closest to the latter and the fact that this X_i converges to θ^n at rate n (in fact, $n\,|X_i - \theta^n|$ converges to an exponential distribution), it follows that n must be the non-trivial rate of convergence.

Asymptotic distribution: $\alpha_n = o(n)$

- Let $\hat{h}_n \equiv n^{1/3}\alpha_n^{2/3}\,(\hat{\theta}^n - \theta^0)$.
- Introducing a local variable and rescaling the criterion function, one gets:

$$\begin{aligned}\hat{h}_n &= \arg\min_h\; n^{2/3}\alpha_n^{1/3}\,\mathbb{P}_n[(Y-(\alpha+\beta)/2)\,(1(X\le\theta^n+h\,n^{-1/3})-1(X\le\theta^n))]\\ &\equiv I_n + II_n\,,\end{aligned}$$

with cb

$$II_n = n^{2/3}\alpha_n^{1/3}\,[M_n(\theta^n + h\,n^{-1/3}\,\alpha_n^{-2/3}) - M_n(\theta^n)]$$

and

$$I_n = \sqrt{n}\,(\mathbb{P}_n - P_n)\,g_{n,h}(x,y)$$

with

$$g_{n,h}(x,y) = n^{1/6}\alpha_n^{1/3}\,[(y-(\alpha+\beta)/2)[1(x\le\theta^n+h\,n^{-1/3})-1(X\le\theta^n))]\,.$$

Next,

$$II_n \to \frac{1}{2}\,h^2\,p_X(\theta_0)\,f'(\lambda_0)\,,$$

and

$$I_n \to_d \sqrt{\sigma^2\,p_X(\theta_0)}\,W(h)\,.$$

- Conclude that $\hat{h}^n \to \arg\min_h\,(\sqrt{\sigma^2\,p_X(\theta_0)}\,W(h) + \frac{1}{2}\,h^2\,p_X(\theta_0)\,f'(\lambda_0))\,.$

Asymptotic distribution: $\alpha_n = n$

- Let $\{\nu^+(h) : h \geq 0\}$ be a homogeneous Poisson process on $[0, \infty)$ with RCLL sample paths and rate parameter $p_X(\theta_0)$.
- Let $\{\epsilon_i\}_{i=1}^{\infty}$ be i.i.d. ϵ and distributed independently of $\nu^+(h)$.
- Let S_i denote the time to the i'th arrival for the Poisson process ν^+, i.e. $S_i = T_1 + T_2 + \ldots + T_i$, where $\{T_j\}_{j=1}^{\infty}$ are the i.i.d. exponential $p_x(\theta_0)$ inter-arrival times corresponding to $\nu^+(h)$.
- For $h \geq 0$, define:

$$\Lambda(h) = \sum_{j=0}^{\nu^+(h)} \left(\epsilon_j + f(\lambda_0 + S_j) - f(\lambda_0)\right) .$$

Asymptotic distribution: $\alpha_n = n$

- To define the process for $h \leq 0$, consider $\nu^-(h)$, an LCRR homogeneous Poisson process on $[0, \infty)$ with parameter $p_X(\theta_0)$ and $\{\tilde{\epsilon}_i\}_{i=1}^{\infty}$ i.i.d. ϵ again, and independent of $\nu^-(h)$.
- Let $\tilde{S}_i$ denote the time to the i'th arrival for the process ν^-.
- For $h \leq 0$, define:

$$\Lambda(h) = \sum_{j=0}^{\nu^-(-h)} \left(-\tilde{\epsilon}_j + f(\lambda_0) - f(\lambda_0 - \tilde{S}_j) \right) .$$

- It can be checked that the process $\Lambda(h)$ thus defined is stationary and has independent increments.
- $V_n(h) \equiv \sum_{i=1}^{n} [Y_i - (\alpha+\beta)/2] [1(X_i \leq \theta^n + h/n) - 1(X_i \leq \theta^n)]$ converges in distribution to $\Lambda(h)$.
- Conclude that $n(\hat{\theta}_n - \theta^n) = \arg\min_h V_n(h) \Rightarrow_d \arg\min_h \Lambda(h)$.

$\alpha_n = n$: DEPENDENCE OF ASYMPTOTICS ON f

- Note that

$$V_n(h) \equiv \sum_{i=1}^{n} [Y_i - (\alpha+\beta)/2]\,[1(X_i \le \theta^n + h/n) - 1(X_i \le \theta^n)]$$

.

- Calculate $E(e^{itV_n(h)}) = \left(1 + \frac{\xi_n}{n}\right)^n$, where

$$\begin{aligned}
\xi_n &= n \int_{\theta^n}^{\theta^n + h/n} [\phi_\epsilon(t)\, \exp[it\{f(n(x-\theta_0)) - f(\lambda_0)\}] - 1]\; p_X(x)\, dx \\
&= \int_0^h [\phi_\epsilon(t)\, \exp[it\{f(n\,(\theta_n + u/n - \theta_0)) - f(\lambda_0)\}] - 1]\; p_X(\theta_0 + u/n)\, du \\
&= \int_0^h [\phi_\epsilon(t)\, \exp[it\{f(\lambda_0 + u) - f(\lambda_0)\}] - 1]\; p_X(\theta_0)\, du + o(1) \\
&\to \phi_\epsilon(t) p_X(\theta_0)\, v(h,t) - p_X(\theta_0)\, h\,,
\end{aligned}$$

where $v(h,t) = \int_0^h \exp[it(f(\lambda_0 + u) - f(\lambda_0))]du\,.$

- Conclude that

$$E(e^{itV_n(h)}) \to \exp\{\phi_\epsilon(t) p_X(\theta_0)\, v(h,t) - p_X(\theta_0)\, h\}\,.$$

$n = o(\alpha_n)$: ASYMPTOTIC DISTRIBUTION

- Define:

$$\tilde{\Lambda}(h) = \sum_{j=0}^{\nu^+(h)} [(1 - (\alpha + \beta)/2) + \epsilon_i]\, 1(h \geq 0)$$

$$+ \sum_{j=0}^{\nu^-(-h)} [(\alpha + \beta)/2 - \tilde{\epsilon}_i]\, 1(h < 0)\,.$$

- Then:

$$\hat{h}_n \equiv n\,(\hat{\theta}_n - \theta^n) = \arg\min_h V_n(h) \Rightarrow_d \arg\min_h \tilde{\Lambda}(h)\,.$$

$n = o(\alpha_n)$: LACK OF DEPENDENCE OF ASYMPTOTICS ON f

- Note that

$$V_n(h) \equiv \sum_{i=1}^{n} [Y_i - (\alpha+\beta)/2]\,[1(X_i \le \theta^n + h/n) - 1(X_i \le \theta^n)]\,.$$

- Calculate $E(e^{itV_n(h)}) = \left(1 + \frac{\xi_n}{n}\right)^n$, where

$$\begin{aligned}
\xi_n &= n \int_{\theta^n}^{\theta^n + h/n} [\phi_\epsilon(t)\, \exp[it\{f(\alpha_n(x-\theta_0)) - f(\lambda_0)\}] - 1]\; p_X(x)\, dx \\
&= \int_0^h [\phi_\epsilon(t)\, \exp[it\{f(\alpha_n(\theta_n + u/n - \theta_0)) - f(\lambda_0)\}] - 1]\; p_X(\theta_0 + u/n)\, du \\
&= \int_0^h [\phi_\epsilon(t)\, \exp[it\{f(\lambda_0 + (\alpha_n u/n)) - f(\lambda_0)\}] - 1]\; p_X(\theta_0)\, du + o(1) \\
&\to \phi_\epsilon(t) p_X(\theta_0)\, v(h,t) - p_X(\theta_0)\, h\,,
\end{aligned}$$

where

$$v(h,t) = \int_0^h \exp[it(1 - f(\lambda_0))] du\; p_X(\theta_0) = \exp[it(1 - (\alpha+\beta)/2))] p_X(\theta_0)\, h\,.$$

- Conclude that

$$E(e^{itV_n(h)}) \to \exp[\phi_\epsilon(t) p_X(\theta_0)\, v(h,t) - p_X(\theta_0)\, h]$$

Bias issues

- For $\alpha_n = o(n)$, the rate of convergence of $\hat{\theta}_n$ for θ^n is $n^{1/3}\alpha_n^{2/3}$ while the rate of convergence of θ^n to θ_0 is slower, namely α_n. So $n^{1/3}\alpha_n^{2/3}(\hat{\theta}_n - \theta_0)$ has an asymptotic bias going to ∞.
- For $\alpha_n = n$ the order of the bias of θ^n for θ_n is the same as the rate of convergence of $\hat{\theta}_n$ to θ^n, namely n, so that $n(\hat{\theta}_n - \theta_0)$ is distributed as $\arg\min_h \Lambda(h) + \lambda_0$.
- For $n = o(\alpha_n)$, the bias reduces at a faster rate than the rate of convergence which is still n and $n(\hat{\theta}_n - \theta_0)$ is distributed as $\arg\min_h \tilde{\Lambda}(h)$.

Asymptotics under the NULL (stump) model

- Suppose that the true model is the NULL model: $f_0(x) = 1(x > \theta_0)$. This is the limit of the models $f_n(x) = f(\alpha_n(x - \theta_0))$.
- Propose $\hat{\theta}_n = \arg\min_\theta \mathbb{P}_n[(Y - (\alpha + \beta)/2)\, 1(X \leq \theta)]$ as an estimate of $\theta_0 = \arg\min_\theta P[(Y - (\alpha + \beta)/2)\, 1(X \leq \theta)]$.
- Not difficult to show that $n(\hat{\theta}_n - \theta_0)$ converges to $\arg\min_h \tilde{\Lambda}(h)$.
- The asymptotic distribution is the exact same as when α_n converges faster than n.

Analogies with parametric models

- Consider regular parametric model $\{p(x, \theta)\}$ and let $X_1, X_2, \ldots, X_n$ i.i.d. θ_{true}.

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- With alternatives converging faster than $\sqrt{n}$, say $\theta_n = \theta_0 + h\, n^{-\gamma}$ with $\gamma > 1/2$, the limit of $\sqrt{n}(\hat{\theta} - \theta_0)$ continues to be the same as under the null.

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- Similarly, in the change-point model, with alternatives converging faster than n, the rate of convergence under the null, the limit distribution is the same as under the null model.
- In the parametric model, with $\theta_n = \theta_0 + n^{-\gamma}$ and $\gamma < 1/2$, the limit distribution of $\sqrt{n}\,(\hat{\theta}_n - \theta_0)$ has an asymptotic bias going to ∞. As we have seen, a similar thing happens with $\alpha_n = o(n)$ in the change-point model.

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- Similarly, in the change-point model, when $\alpha_n = n$ (the rate of the change-point estimate under the null model), f, which can be interpreted as the direction of approach of the continuous models to the stump, figures in the limit.
- Note that the rate of convergence of the estimator in the change-point model depends on α_n, the rate at which the alternatives approach the null. This is however not the case for parametric models.

Connections between the limit distributions

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- Suppose $n = o(\alpha_n)$. Then $c = c(n) \to \infty$ and it appears that (heuristics for now) Λ_{f_c} approaches $\tilde{\Lambda}$. Also, λ_{0,f_c} approaches 0. This is reassuring.

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- When $\alpha_n = o(n)$, calculations reveal that the approximation will work correctly, provided $c_n^{2/3} \arg\min_h \Lambda_{f_{c_n}}(h)$ converges to the limit distribution of $n^{1/3}\alpha_n^{2/3}(\hat{\theta}_n - \theta^n)$. Does this happen? Now $c_n \to 0$.

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- Compute $\hat{\theta}_n$ and approximate the distribution of $n(\hat{\theta}_n - \theta_0)$ by that of $\arg\min_h \Lambda_{f_c}(h) + \lambda_{0,f_c}$.
- Is this approximation consistent with what we would get if we used the correct limit distribution as an approximation?
- Suppose $n = o(\alpha_n)$. Then $c = c(n) \to \infty$ and it appears that (heuristics for now) Λ_{f_c} approaches $\tilde{\Lambda}$. Also, λ_{0,f_c} approaches 0. This is reassuring.
- When $\alpha_n = o(n)$, calculations reveal that the approximation will work correctly, provided $c_n^{2/3} \arg\min_h \Lambda_{f_{c_n}}(h)$ converges to the limit distribution of $n^{1/3}\alpha_n^{2/3}(\hat{\theta}_n - \theta^n)$. Does this happen? Now $c_n \to 0$.
- The possibility of conducting inference based on these ideas needs to be investigated.

The three parameter problem

- In this problem, we compute:

$$(\hat{\gamma}_n, \hat{\delta}_n, \hat{\theta}^n) = \arg\min_{\gamma,\delta,\theta} \mathbb{P}_n[(Y-\gamma)^2 1(X \le \theta) + (Y-\delta)^2 1(X > \theta)]\,.$$

- Population parameters at stage n:

$$(\gamma_n, \delta_n, \theta^n) = \arg\min_{\gamma,\delta,\theta} P_n[(Y-\gamma)^2 1(X \le \theta) + (Y-\delta)^2 1(X > \theta)]\,.$$

- Under mild conditions:

$$(\alpha_n|\theta^n - \theta_0|, \sqrt{\alpha}_n|\gamma_n - \gamma_0|, \sqrt{\alpha}_n|\delta_n - \delta_0|) = O(1)\,.$$

The three parameter problem

Under mild assumptions

- When $\alpha_n = o(n)$,

$$\{n^{1/3}\alpha_n^{2/3}|\hat{\theta}^n - \theta^n|, n^{1/3}\alpha_n^{1/6}|\hat{\gamma}_n - \gamma_n|, n^{1/3}\alpha_n^{1/6}|\hat{\delta}_n - \delta_n|\} = O_P(1)\,.$$

- When $\alpha_n = n$,

$$\{n|\hat{\theta}^n - \theta^n|, \sqrt{n}|\hat{\gamma}_n - \gamma_n|, \sqrt{n}|\hat{\delta}_n - \delta_n|\} = O_P(1)\,.$$

- When $n = o(\alpha_n)$,

$$\{n|\hat{\theta}^n - \theta^n|, \sqrt{n}|\hat{\gamma}_n - \gamma_n|, \sqrt{n}|\hat{\delta}_n - \delta_n|\} = O_P(1)\,.$$