Midterm 2 : Stat 426

12/3/2004

Solution :

1. (a) $\boxed{F}$ If $\alpha > \gamma$, the CI will be smaller.

1. (b) $\boxed{F}$ The CI will be from $-\infty$ to $+\infty$ and will be of little practical value.

1. (c) $\boxed{F}$ The log-likelihood $f\underline{n}$ may not be differentiable.

1. (d) $\boxed{T}$ Law of large ~~nu~~ numbers.

1. (e) $\boxed{T}$ $\sum_{i=1}^{n} (X_i-\mu)^2/\sigma^2 \sim \chi_n^2$ - this can be used as a pivot.

2.

2. $Y_i = \beta X_i + \epsilon_i$, ϵ_i 's are i.i.d $N(0, \sigma^2)$

(a) $E(Y_i) = \beta X_i + E(\epsilon_i) = \beta X_i$

$Var(X_i) = Var(\epsilon_i) = \sigma^2$

$Y_i \sim N(\beta X_i, \sigma^2)$

– independent but not identically distributed (different means)

(b) The joint density can be written as

$$\prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(Y_i - \beta X_i)^2\right]$$

$$= \frac{1}{\sigma^n (2\pi)^{n/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \beta X_i)^2\right]$$

Thus, the log likelihood (call it $l(\beta, \sigma)$) is –

$$l(\beta, \sigma) = -n \log \sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \beta X_i)^2 + \text{constant}$$

$$\Rightarrow \frac{\partial L}{\partial \beta} = -\frac{1}{2\sigma^2} \cdot 2 \sum_{i=1}^{n}(Y_i - \beta X_i) \cdot (-X_i)$$

equating with 0,

$$\hat{\beta} = \frac{\sum X_i Y_i}{\sum X_i^2}$$

$$\frac{\partial L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(Y_i - \beta x_i)^2$$

equating with 0 and plugging in $\hat{\beta}$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{\beta} x_i)^2$$

$$\therefore \hat{\sigma} = \left[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{\beta} x_i)^2\right]^{1/2}$$

where $\hat{\beta} = \dfrac{\sum x_i y_i}{\sum x_i^2}$

Then $(\hat{\beta}, \hat{\sigma})$ is the MLE for β, σ.

(3) X_1, X_2 are iid $Exp(\lambda)$

Thus, $2\lambda X_1, 2\lambda X_2$ are iid $Exp(1/2)$ or χ^2_2

Thus, $2\lambda(X_1+X_2) \sim \chi^2_4$

Thus,

$$P\left[\chi^2_{4,\alpha/2} \le 2\lambda(X_1+X_2) \le \chi^2_{4,(1-\alpha/2)}\right] = 1-\alpha$$

Thus, the required CI is,

$$\left[\frac{\chi^2_{4,\alpha/2}}{2(X_1+X_2)}, \frac{\chi^2_{4,(1-\alpha/2)}}{2(X_1+X_2)}\right]$$