# Supplementary Final Exam: Statistics 426.

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**Announcement:** The exam carries 80 points but the maximum you can score is 70. Half of your score on the final adds towards your final grade in this course.

## 1 Section A

**This section carries 20 points. There are 5 statements. You need to judge whether each statement is true or false and provide adequate justification. Answering this section is mandatory.**

(1) Consider an urn with 100 balls in it, numbered 1 through 100. The chance that you never draw Ball 1 in $n$ draws from the urn with replacement goes to 1 as $n$ goes to $\infty$.

(2) For large $n$, a Poisson($n$) distribution can be well–approximated by a normal distribution.

(3) If $T$ follows a $t$ distribution, then $T^2$ follows an $F$ distribution.

(4) Suppose that an investigator is asked to choose 50 people to interview out of a population of size 1000 to judge their political affiliation, Republic or Democrat. The investigator reports $\hat{p} = 0.9$ as an estimate of $p$, the proportion of Republicans in the population, based on his sample. Then, a reasonable estimate for $p$ (ignoring the finite population correction) is

$$\left[\hat{p} - \frac{\hat{p}\,(1-\hat{p})}{\sqrt{n-1}}\, z_{1-\alpha/2}\ \ ,\ \ \hat{p} + \frac{\hat{p}\,(1-\hat{p})}{\sqrt{n-1}}\, z_{1-\alpha/2}\right].$$

(5) The best confidence intervals are typically those that have confidence level 1 since we are absolutely certain that they contain the true parameter of interest. ( $4 \times 5 = 20$ points)

## 2 Section B

**This section carries 60 points. There are 4 questions, each worth 20 points and you need to answer 3 out of 4.**

(1) (a). Let $Y$ follow $\text{Exp}(\lambda)$ and for $k = 1, 2, \ldots,$ define

$$X = k \text{ if } k - 1 < Y \leq k\,.$$

Show that $E\,X = \frac{e^{\lambda}}{e^{\lambda}-1}$ and $\text{Var}(X) = \frac{e^{\lambda}}{(1-e^{\lambda})^2}$ . **Hint:** What is the distribution of $X$?

(b) Let $X$ and $Y$ be i.i.d. Exponential($\lambda$) random variables and define $U = X + Y$ and $V = X/(X+Y)$. Compute the joint density of $(U, V)$ and identify the marginal densities of $U$ and $V$. Are $U$ and $V$ independent? (7 + 13 = 20 points)

(2) (a) Let $X_1, X_2, \ldots, X_n$ be i.i.d. $U(0, \theta)$ $(\theta > 0)$. Let $X_{(n)} = \max(X_1, X_2, \ldots, X_n)$. Let $W = X_{(n)}/\theta$.

Show that the density function of $W$ is,

$$f_W(w) = n\,w^{n-1} \quad , \quad 0 < w < 1\,.$$

(**See Section 3.7, Page 100 of Rice, for discussion on the distribution of the maximum.**) Hence conclude that

$$\text{Prob}_{\theta}\left[\left(\frac{\alpha}{2}\right)^{1/n} \leq \frac{X_{(n)}}{\theta} \leq \left(1 - \frac{\alpha}{2}\right)^{1/n}\right] = 1 - \alpha\,.$$

Use the above result to find a level $1 - \alpha$ confidence interval for $\theta$.

(b) In a city 30% of people are conservatives, 50% are liberals and 20% are independents. In a particular election, 60% of conservatives voted, 80 % of liberals voted and 50% of independents voted. If a person is selected at random from the population and it is learnt that they did not vote in the last election, what is the chance that the randomly chosen person is a liberal ?

(c) Suppose that the diameter of a circle is a random variable having the following probability density function.

$$f(x) = \frac{1}{8}\,(3x + 1), \quad 0 < x < 2$$

and 0 otherwise. Determine the probability density function of the area of the circle. (8 + 6 + 6 = 20 points)

(3) Let $X_1, X_2, \ldots, X_n$ be i.i.d. from the density,

$$f(x, \theta) = \frac{x}{\theta^2}\, e^{-\frac{x^2}{2\,\theta^2}} \quad , \quad x > 0 \, , \; \theta > 0 \, .$$

(a) Show that $E(X_1^2) = 2\,\theta^2$ and use this to compute a MOM estimate of $\theta$. (Hint: What is the distribution of $X_1^2$?)

(b) Compute the MLE of $\theta$ and show that it coincides with the MOM in (a).

(c) By the CLT, $\sum_{i=1}^n X_i^2/n$ is approximately normal with parameters $(\mu(\theta), \sigma^2(\theta))$, where $\mu(\theta)$ and $\sigma^2(\theta)$ are functions of $\theta$. Find $\mu(\theta)$ and $\sigma^2(\theta)$ and subsequently find a level $1 - \alpha$ confidence interval for $\theta$. (7 + 6 + 7 = 20 points)

(4) Consider the following testing problem: $X_1, X_2, \ldots, X_n$ are i.i.d. Exp$(\lambda)$

(i) Write down $f_1(X_1, X_2, \ldots, X_n)$, the joint density of the data under the hypothesis that $\lambda = \lambda_1$ and $f_0(X_1, X_2, \ldots, X_n)$, the joint density under the hypothesis that $\lambda = \lambda_0$, where $\lambda_1 < \lambda_0$ .

(ii) Show that $\log\left(f_1(X_1, X_2, \ldots, X_n)/f_0(X_1, X_2, \ldots, X_n)\right)$ is an increasing function of $\overline{X}_n$.

(iii) Suppose $c_{\alpha,n}$ is such that $P_{\lambda_0}\left(\overline{X}_n \geq c_{\alpha,n}\right) = \alpha$. Show that $c_{\alpha,n} = q_{2n}(1-\alpha)/2\,\lambda_0\, n$ where $q_k(\beta)$ is the $\beta$'th quantile of the $\chi_k^2$ distribution.

(iv) Show that for any $\lambda$,

$$\beta(\lambda) = P_\lambda\left(\overline{X}_n \geq c_{\alpha,n}\right) = 1 - F_{2n}\left(\frac{\lambda}{\lambda_0}\, q_{2n}(1-\alpha)\right)$$

where $F_{2n}$ is the distribution function of $\chi_{2n}^2$ and conclude that this is decreasing in $\lambda$ (and hence increasing in $1/\lambda$).

(20 points).