

# Midterm 1 – Statistics 426.

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**Announcement:** This exam is open book, open notes and carries 24 points, but the maximum that you can score is 20. Problem 2(iv) is more difficult than the others and I would advocate trying it last.

- 1 Let  $X_1$  and  $X_2$  be i.i.d. standard normal random variables (a standard normal random variable is normal with mean 0 and variance 1).

(i) Write down the joint density  $f(x_1, x_2)$  of  $(X_1, X_2)$ .

(ii) Define

$$W_1 = \frac{\sqrt{3} X_1 + X_2}{2} \quad \text{and} \quad W_2 = \frac{X_1 - \sqrt{3} X_2}{2}.$$

Compute the joint density of  $(W_1, W_2)$  and the marginal densities of  $W_1$  and  $W_2$ ? What is the probability that  $W_1^2 + W_2^2 < 1$ ? (2 + (5+2+2) = 11 points).

- 2 Consider an urn that contains three balls marked 1, 2 and 3. I pick a ball out of the urn at random. Let  $X_1$  denote the number on this ball. Without returning this ball back to the urn, I draw another ball from the urn, again at random. Let  $X_2$  denote the number on this ball. There is now one ball remaining in the urn and I pick it out. Let  $X_3$  denote the number on this ball.

(i) Clearly, I always know the value of  $X_3$  even before picking the third ball out of the urn, if I have noted the numbers  $X_1$  and  $X_2$ . Why is then  $X_3$  still a random variable?

(ii) Compute  $P(X_1 = 1, X_2 = 2, X_3 = 3)$ . What is the value of  $P(X_1 = i, X_2 = j, X_3 = k)$ , where  $(i, j, k)$  is some permutation of  $(1, 2, 3)$ ?

(iii) Compute  $E(X_1 + X_2 + X_3)$  and  $\text{Var}(X_1 + X_2 + X_3)$ .

(iv) It is not difficult to show that  $X_1$ ,  $X_2$  and  $X_3$  have the same marginal distributions and that the joint distributions of  $(X_1, X_2)$ ,  $(X_2, X_3)$  and  $(X_1, X_3)$  are the same. Using

this result and the value of  $\text{Var}(X_1 + X_2 + X_3)$  obtained in (iii) (or otherwise) show that  $\text{Cov}(X_1, X_2) < 0$ . ( $3 + 3 + 3 + 4 = 13$ )

**Helpful results.**

(a) Recall that

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2|A_1) P(A_3|A_1 \cap A_2) .$$

(b) Also recall that,

$$\text{Var} \left( \sum_{i=1}^n X_i \right) = \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{i < j} \text{Cov}(X_i, X_j) .$$