Midterm 1 Solution Sketch: Stat 426.

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(1) (i) The joint density of (X_1, X_2) is,

$$f(x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{x_1^2 + x_2^2}{2}\right) .$$

(ii) To compute the joint density of (W_1, W_2) we need to express (X_1, X_2) as a function of (W_1, W_2). We have,

$$2\,W_1 = \sqrt{3}\,X_1 + X_2 \;\text{ and }\; 2\,W_2 = X_1 - \sqrt{3}\,X_2 .$$

Therefore,

$$2\sqrt{3}\,W_1 + 2\,W_2 = 3\,X_1 + \sqrt{3}\,X_2 + X_1 - \sqrt{3}\,X_2 = 4\,X_1 ,$$

showing that

$$X_1 = \frac{\sqrt{3}\,W_1 + W_2}{2} .$$

Similar manipulations, or direct substitution now gives,

$$X_2 = \frac{W_1 - \sqrt{3}\,W_2}{2} .$$

The joint density of (W_1, W_2) is

$$f_W(w_1, w_2) = \frac{1}{2\pi} \exp\left(-\frac{(\sqrt{3}\,w_1 + w_2)^2/4 + (w_1 - \sqrt{3}\,w_2)^2/4}{2}\right) J ,$$

where

$$J = \left|\frac{\partial X_1}{\partial W_1}\frac{\partial X_2}{\partial W_2} - \frac{\partial X_1}{\partial W_2}\frac{\partial X_2}{\partial W_1}\right| .$$

Check that $J = 1$ and that

$$(\sqrt{3}\,w_1 + w_2)^2/4 + (w_1 - \sqrt{3}\,w_2)^2/4 = w_1^2 + w_2^2 .$$

It follows that

$$f_W(w_1, w_2) = \frac{1}{2\,\pi} \exp\left(-\frac{w_1^2 + w_2^2}{2}\right) = \frac{1}{\sqrt{2\,\pi}} \exp\left(-\frac{w_1^2}{2}\right) \frac{1}{\sqrt{2\,\pi}} \exp\left(-\frac{w_2^2}{2}\right).$$

It follows that W_1 and W_2 are i.i.d. $N(0,1)$ random variables.

The chance that $W_1^2 + W_2^2 < 1$ is simply the chance that a χ_2^2 random variable is less than 1. But a χ_2^2 is just an exponential 1/2 random variable. Hence this probability can be directly evaluated. (How?)

(2) (i) The value of X_3 is known only conditionally on the values of X_1 and X_2, but not otherwise. Note that $X_3 = 6 - (X_1 + X_2)$; since $X_1 + X_2$ is a non-degenerate random variable, so clearly is X_3 and $P(X_3 = i) = P(X_1 + X_2 = 6 - i)$.

(ii)

$$\begin{aligned} P(X_1 = i, X_2 = j, X_3 = k) &= P(X_1 = i)\, P(X_2 = j | X_1 = i)\, P(X_3 = k | X_1 = i, X_2 = j) \\ &= \frac{1}{3} \times \frac{1}{2} \times 1 \\ &= \frac{1}{6}. \end{aligned}$$

(iii) Note that $X_1 + X_2 + X_3 = 6$ identically, i.e. with probability 1. Thus $E(X_1 + X_2 + X_3) = 6$ and $\mathrm{Var}(X_1 + X_2 + X_3) = 0$.

(iv)

$$\mathrm{Var}(X_1 + X_2 + X_3) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{i<j} \mathrm{Cov}(X_i, X_j).$$

Since the X_i's have the same marginal distributions and the pairs (X_1, X_2), (X_1, X_3) and (X_2, X_3) are identically distributed we have,

$$\mathrm{Var}(X_1 + X_2 + X_3) = 3\,\mathrm{Var}(X_1) + 6\,\mathrm{Cov}\,(X_1, X_2).$$

But $\mathrm{Var}(X_1 + X_2 + X_3) = 0$ showing that,

$$3\,\mathrm{Var}(X_1) + 6\,\mathrm{Cov}\,(X_1, X_2) = 0$$

which shows that,

$$-\mathrm{Var}(X_1) = 2\,\mathrm{Cov}\,(X_1, X_2).$$

Hence $\mathrm{Cov}\,(X_1, X_2) < 0$.