Midterm 1: Stat 426.

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Announcement: The total number of points is 25 but the maximum you can score is 20.

- 1. Alvie Singer lives at O in the diagram below and has four friends who live at A, B, C and D. One day Alvie decides to go visiting, so he tosses a fair coin twice to decide which of the four friends to visit. Once at a friend's house, he will either return home or else proceed to one of the two adjacent houses (such as O, A or C when at B) with each of the three possibilities having probability 1/3. In this way, Alvie continues to visit friends till he returns home. Let X be the number of times that Alvie visits a friend. Find the probability mass function of X, i.e. $P(X = j)$ for $j = 0, 1, 2, \ldots$ etc. (7 points)

  Solution: Note that since Alvie visits at least one friend, $P(X = 0) = 0$. The chance that Alvie visits exactly one friend, $P(X = 1)$, is simply 1/3, since the chance that Alvie returns home after visiting the first friend is 1/3 (with probability 2/3 he goes to an adjacent friend's house). In general, the chance that Alvie visits exactly j friends is

  $$P(X = j) = (2/3)^{j-1}\,(1/3)\,,$$

  since this happens if Alvie moves from the 1'st friend's house to the 2'nd friend's house (with probability 2/3), from the 2'nd to the 3'rd(again with probability 2/3) and so on till he reaches the j'th friend (in $j-1$ steps) and then comes back home from the j'th friend's house (with probability 1/3).

- 2. Let X_1 and X_2 be independent $N(0,1)$ random variables. Let

  $$W_1 = \frac{X_1 + X_2}{\sqrt{2}} \quad \text{and} \quad W_2 = \frac{X_1 - X_2}{\sqrt{2}}\,.$$

  (i) Using the Jacobian Theorem (or otherwise) show that W_1 and W_2 are independent $N(0,1)$ random variables.

  Solution: Now,

  $$\sqrt{2}\,W_1 = X_1 + X_2 \ \text{and} \ \sqrt{2}\,W_2 = X_1 - X_2\,,$$

showing that

$$\sqrt{2}\,(W_1 + W_2) = 2\,X_1 \text{ and } \sqrt{2}\,(W_1 - W_2) = 2\,X_2\,.$$

It follows that

$$X_1 = g_1(W_1, W_2) = \frac{W_1 + W_2}{\sqrt{2}} \text{ and } X_2 = g_2(W_1, W_2) = \frac{W_1 - W_2}{\sqrt{2}}\,.$$

Now,

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\,\pi}\exp\left(-\frac{x_1^2 + x_2^2}{2}\right)\,.$$

Using the Jacobian theorem, the joint density of (W_1, W_2) is simply

$$f_{W_1,W_2}(w_1, w_2) = \frac{1}{2\,\pi}\exp\left(-\frac{g_1(w_1,w_2)^2 + g_2(w_1,w_2)^2}{2}\right) \times J_{g_1,g_2}(w_1, w_2)\,,$$

where

$$J_{g_1,g_2}(w_1, w_2) = \left|\frac{\partial\, g_1}{\partial\, w_1}\,\frac{\partial\, g_2}{\partial\, w_2} - \frac{\partial\, g_2}{\partial\, w_1}\,\frac{\partial\, g_1}{\partial\, w_2}\right|\,.$$

Now,

$$\frac{\partial\, g_1}{\partial\, w_1} = \frac{1}{\sqrt{2}} = -\frac{\partial\, g_2}{\partial\, w_2}$$

and

$$\frac{\partial\, g_2}{\partial\, w_1} = \frac{\partial\, g_1}{\partial\, w_2} = \frac{1}{\sqrt{2}}$$

showing that

$$J_{g_1,g_2}(w_1, w_2) = 1\,.$$

Also,

$$g_1(w_1, w_2)^2 + g_2(w_1, w_2)^2 = \left(\frac{w_1 + w_2}{\sqrt{2}}\right)^2 + \left(\frac{w_1 - w_2}{\sqrt{2}}\right)^2 = w_1^2 + w_2^2\,.$$

It follows that,

$$f_{W_1,W_2}(w_1, w_2) = \frac{1}{2\,\pi}\exp\left(-\frac{w_1^2 + w_2^2}{2}\right) = \frac{1}{\sqrt{2\,\pi}}\exp\left(-\frac{w_1^2}{2}\right) \times \frac{1}{\sqrt{2\,\pi}}\exp\left(-\frac{w_2^2}{2}\right)\,.$$

Hence W_1 and W_2 are i.i.d. $N(0,1)$ random variables.

(ii) How is W_2^2 related to $(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2$? ($\bar{X}$ is just the average of X_1 and X_2)

Solution: Note that,

$$X_1 - \overline{X} = X_1 - \frac{X_1 + X_2}{2} = \frac{X_1 - X_2}{2} = -(X_2 - \overline{X})\,,$$

so that

$$(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 = \frac{(X_1 - X_2)^2}{2} = W_2^2 \,.$$

(iii) Deduce from (ii) that $(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2$ follows a χ_1^2 distribution. Also, show that it is independent of $\bar{X}$.

Solution: Since W_2 is $N(0, 1)$, it follows that W_2^2 is a χ_1^2 random variable. Also W_2 is independent of W_1 and therefore, so is W_2^2. It follows that $W_1/\sqrt{2} = \overline{X}$ is independent of $W_2^2 = (X_1 - \bar{X})^2 + (X_2 - \bar{X})^2$.

(iv) Identify the distribution of $(X_1 + X_2)/ \mid X_1 - X_2 \mid$. You do not need to work out the density; in fact, don't go down that track. Just identify it as a distribution you know/have seen. (8 + 3 + 4 + 3 = 18 points)

Solution: We have

$$\frac{(X_1 + X_2)}{\mid X_1 - X_2 \mid} = \frac{(X_1 + X_2)/\sqrt{2}}{\mid X_1 - X_2 \mid /\sqrt{2}} = \frac{W_1}{\mid W_2 \mid} = \frac{W_1}{\sqrt{W_2^2/1}}$$

showing that the ratio follows a t distribution on 1 degree of freedom. This is also the Cauchy distribution.