A Note on the Exponential Distribution

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The exponential distribution is an example of a continuous distribution. A random variable X is said to follow the exponential distribution with parameter λ if its distribution function F is given by: $F(x) = 1 - e^{-\lambda x}$ for $x > 0$. Recall that the distribution function $F(x) = P(X \leq x)$ by definition and is an increasing function of x. Since $F(0) = 0$, it follows that X is bigger than 0 with probability 1.

The exponential distribution is often used to model the failure time of manufactured items in production lines, say, light bulbs. If X denotes the (random) time to failure of a light–bulb of a particular make, then the exponential distribution postulates that the probability of survival of the bulb decays exponentially fast – to be precise, $P(X > x) = e^{-\lambda x}$. Notice that the bigger the value of λ, the faster the decay. This indicates that for large λ the average time of failure of the bulb is smaller. This is indeed true. It is not difficult to check (verify this) that $E(X) = \lambda^{-1}$.

An interesting feature of the exponential distribution is its memoryless property (discussed in the Probability Refresher Notes) – the distribution "forgets" the past. In equations:

$$P(X > x + y \mid X > x) = P(X > y)\,.$$

Given that a bulb has survived x units of time, the chance that it survives a further y units of time is the same as that of a fresh bulb surviving y units of time. In other words, past history has no effect on the bulb's performance. This is of course not realistic for many applications. However, if an item is typically known to "live" a long time from start, and x and y are not too big, the memoryless property may not be too outlandish to assume. In other words, if an item has not been used that much, it is almost as good as new. The memoryless property of the exponential is easy to verify from definitions (try this).

The "memorylessness" of the exponential distribution is reflected in the fact that the *instantaneous failure rate* for the exponential distribution is constant as a function of time. This quantity $\lambda(t)$ is defined in the following manner. Consider the chance that the bulb fails in the interval $(t, t + dt)$ where dt is a small interval, given that it has survived till time t. This conditional probability is simply $P(X \in (t, t + dt) \mid X > t)$. This can be written as:

$$P(X \in (t, t + dt) \mid X > t) = \lambda(t)\, dt + o(dt) \quad (**)$$

where

$$\lambda(t) \equiv \lim_{dt \to 0} \frac{P(X \in (t, t+dt) \mid X > t)}{dt}$$

and the term $o(dt)$ indicates a quantity that satisfies $o(dt)/dt \to 0$ as $dt \to 0$.

For *any* random variable that is non–negative with probability 1, the quantity $\lambda(t)$ can be legitimately defined, and can be shown to be given by:

$$\lambda(t) = \frac{f(t)}{1 - F(t)}\,,$$

where f is the density function of the random variable and F is its distribution function. For the exponential distribution, check that $\lambda(t) = \lambda$. Note that a larger value of $\lambda(t)$ implies a greater chance of failing in the instant after t (conditional on surviving till t) by the display $(*\,*)$ (since the second term is of smaller order than dt it can be "ignored"). This is why $\lambda(t)$ is called the *instantaneous failure rate* or the *hazard rate.* For the exponential distribution, the hazard rate does not change with time!! This is certainly not a good model, say, for the time to development of cancer in a living organism. All other factors remaining the same, the propensity for developing cancer is typically seen to increase with age.