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Hedging of Contingent Claims in Incomplete Markets

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1 Introduction

This report surveys important results in the literature on the problem of hedging contingent claims in incomplete markets. Consider a probability space $(\Omega, \mathcal{F}, P)$ and let X be a stochastic process describing the fluctuation of the stock price. Given a contingent claim H, the problem is to find an "optimal" admissible trading strategy, which is a dynamic porfolio of stock and bond (with fixed price), that can almost surely achieve the value H at some terminal time T.

Under the basic assumption of viability (i.e. absence of arbitrages), there exists an equivalent probability measure $P^* \approx P$ under which X is a martingale. This means that X is a *semimartingale*. When the market is complete, i.e., all contingent claims are attainable, the problem is completely solved by Harrison and Kreps [HK79]. In complete markets, it was shown that such a martingale measure is unique. Thus Harrison and Kreps proved that any contingent claim H in a complete market can be expressed as a stochastic integral of X. This allows us to find *self-financing* strategies, i.e, those that essentially incurr no cost. This theory underlies the insights behind the result of option pricing theory starting with the seminal papers by Black and Scholes [BS73] and Merton [M73].

When the market is incomplete, the problem is more delicate. The martingale measure P^* is no longer unique. A general claim is not necessarily a stochastic integral of X, and there may not exist self-financing admissible trading strategy for a given H. An admissible trading strategy in general must incur some *intrinsic risk* of the market.

The first attack on the problem was by Follmer and Sondermann [FS86], who considers a special case in which X is already a martingale under P in an incomplete market. These authors introduced to notion of *risk-minimizing* strategies, which minimizes the risk in a sequential sense. It was shown that such a risk-minimizing admissible strategy always exists. Moreover, it is also *mean-self-financing*, which means the corresponding cost process is a martingale. The optimal strategy can now be determined using Kunita-Watanabe projection technique. In addition, the optimal strategy changes according to an absolutely continuous change of the underlying martingale measure P. We discuss these results in Section 3.

When X is in general a semimartingale, the notion of risk-minimization introduced in [FS86] is not directly applicable. However, Follmer and Schweizer [FS90] extended this notion to risk-minimization in a local sense by considering small perturbations to the strategies. Interestingly, locally risk-minimizing strategies are shown to be also mean-self-financing. From here it is possible to derive the necessary and sufficient conditions for the existence and uniqueness of locally risk-minimizing strategies. These conditions essentially involve a decomposition of contingent claim H into orthogonal components that is somewhat similar to (but not the same as) the Kunita-Watanabe decomposition.

There are two approaches to finding the optimal strategy in the locally risk-minimizing sense. One can characterize the optimal strategy by an *optimality equation* and focus on solving it. This is the approach pursued in [S91]. This approach, along with the notion of locally risk-minimization, is covered in Section 4.

The other approach, which seems more natural, is to use a Girsanov transformation in order to shift the problem back to the space of martingale measure. As mentioned earlier, due to the incompleness of the market, there may be more than one equivalent martingale measure. Hence, the notion of *minimal martingale measure* introduced by Follmer and Schweizer [FS90] to describe the equivalent martingale measure that preserves the structure of the original probability space P as far as possible. These authors showed how to compute the optimal strategy using this change of measure. Section 5 is devoted to these powerful results.

In Section 6, we consider a special case where the incompleteness of the market comes from *incomplete information* [FS90]. The assumption is that the contingent claim H is attainable with respect to a larger filtration that the filtration that the stock price process X actually adapted to. This proves to be a nice application of the theory developed in the previous sections. The claim H can be shown to be decomposed into nice form that allows the existence and uniqueness of optimal strategy, which can be determined by projection technique into the original filtration.

Finally, Section 7 concludes the report with a few remarks.

2 Basic definitions

Let $X = (X_t)_{0 \leq t \leq T}$ be a real-valued stochastic process with continuous path on some probability space $(\Omega, \mathcal{F}, P)$ with right-continuous filtrations $(\mathcal{F}_t)_{0 \leq t \leq T}$ such that $\mathcal{F}_T = \mathcal{F}$. The process X describes the price fluctuation of a given stock.

We assume X to be a *semimartigale* with the Doob-Myer decomposition:

$$X = X_0 + M + A \tag{1}$$

where (1) M is a square-integrable martingale process (cf. [KS88]), (2) A is a predictable process of bounded variation $|A|$. All this amounts to

$$E[X_0^2 + \langle X \rangle_T + |A|_T^2] < \infty \tag{2}$$

Here $\langle X \rangle = \langle M \rangle$ denotes the quadratic variation process of X resp. M (cf. [KS88]).

Let $\overline{P}$ denote the finite measure on $(\Omega \times [0,T], \mathcal{P} = \mathcal{F} \times \mathcal{B}([0,T]))$ given by

$$\overline{P}[A] = E[\int_0^T 1_A(\omega, t) d\langle X \rangle_t(\omega)] \tag{3}$$

A *trading strategy* is of the form $\varphi = (\xi, \eta)$, where $(\xi_t)_t$ and $(\eta_t)_t$ describe the successive amounts invested into the stock and into the bond. Here we assume the bond price to be fixed to 1. Thus,

$$V_t = \xi_t X_t + \eta_t \tag{4}$$

is the *value of the porforlio* at time t. We need several technical assumptions:

Definition 1 *$\varphi = (\xi, \eta)$ is called a strategy if*
(a) ξ is a predicable process, and $\xi \in L^2(\overline{P})$,
(b) η is adapted,
(c) $V = \xi X + \eta$ has right-continuous paths and $V_t \in L^2(P), 0 \leq t \leq T$.

The *accumulated gain* can be computed as $\int_0^t \xi_s dX_s$, $0 \leq t \leq T$. When X is actually a martingale, the accumulated gain process is also a martingale with mean 0 and variance $E \int_0^t \xi_s^2 d\langle X \rangle_s$ at each fixed t.

The *accumulated cost* is defined as

$$C_t = V_t - \int_0^t \xi_s dX_s \tag{5}$$

Definition 2 *A strategy φ is* self-financing *if its cost process $(C_t)_{0\leq t\leq T}$ is time-invariant.*

A *contingent claim* at time T is given by a random variable $H \in L^2(\Omega, \mathcal{F}_T, P)$. We are concerned with only admissible strategies that can achieve H at T, that is, $V_T(\varphi) = H$ P-a.s.

We start with some well-known classic results (cf. [E02]). Under the assumption of viability (i.e there are no arbitrages), there exists a probability measure $P^* \approx P$ such that X is a martingale under P^*. In a complete market, such P^* is unique, and one can determine the unique self-financing φ based soly on measure P^*. Much of the theory of complete market was initiated and developed in the seminal paper by Harrison and Kreps [HK79]. It was a generalization of the economic insights underlying the Black-Scholes formula for pricing European stock option.

When the market is incomplete, the equivalent martingale measure P^* is no longer unique. The problem now is more delicate, because we can imagine that each martingale measure P^* may induce a different admissible strategy. Thus, we need to have a way to choose the optimal one among them.

From an economic viewpoint, the existence of self-financing strategies in complete markets implies that these markets are risk-free. On the other hand, incomplete markets have *intrinsic risk*. The problem is to find a strategy that minimizes this risk, which is to be defined appropriately.

3 Incomplete market: Case $P = P^*$

In this section, we consider a special case in which P is already a martingale measure $P = P^*$. In other words, X is $P-$martingale.[1] This case was considered and completely solved by Follmer and Sondermann [FS86], who first introduced the important notions of *risk-minimizing* strategy and *mean-self-financing* strategy.

3.1 Risk-minimization and Mean-self-financing concept

Consider an H-admissbile strategy $\varphi = (\xi, \eta)$. The *remaining risk* of φ at a fixed time t is defined by

$$R_t(\varphi) = E[(C_T - C_t)^2 | \mathcal{F}_t] \tag{6}$$

A strategy $\tilde{\varphi}$ is called an *admissible continuation* of φ if $\tilde{\varphi}$ coincides with φ al all times $< t$ and $V_T(\tilde{\varphi}) = H$ P-a.s.

Definition 3 *A strategy φ is called* risk-minimizing *if φ at any time minimizes remaining risk. That is, for any $0 \leq t \leq T$,*

$$R_t(\varphi) \leq R_t(\tilde{\varphi})$$

P-a.s. for every admissible continuation $\tilde{\varphi}$ of φ at time t.

Remark. Any self-financing strategy is clearly risk-minimizing since $R_t(\varphi) \equiv 0$. The converse is not true in incomplete markets. We will see why shortly.

Definition 4 *φ is* mean-self-financing *if its corresponding cost process $C = (C_t)_{0\leq t\leq T}$ is a martingle.*

The following lemmas are useful.

[1] The Doob-Myer decomposion becomes merely $X = M$ in this case.

Lemma 5 *φ is mean-self-financing if and only if $(V_t)_{0\le t\le T}$ is a square-integrable martingale.*

Proof: Clear from the definition of the cost and value processes. ■

Lemma 6 *An admissible risk-minimizing strategy is mean-self-financing.*

Proof: Let φ be a risk-minimizing strategy. Fix $0 \le t_0 \le T$. Define $\tilde{\varphi}$ to be an admissible continuation of φ at t_0, such as for $t \ge t_0$,

$$\begin{aligned}
\tilde{\xi}_t &= \xi_t \\
\tilde{\eta}_t &= \tilde{C}_t + \int_0^t \xi_s dX_s - \xi_t X_t \\
\tilde{C}_t &= E[C_T|\mathcal{F}_t]
\end{aligned}$$

Thus,

$$\begin{aligned}
R_{t_0}(\varphi) &= E[(C_T - C_{t_0})^2|\mathcal{F}_{t_0}] \\
&= E[(\tilde{C}_T - \tilde{C}_{t_0})^2|\mathcal{F}_{t_0}] + (\tilde{C}_{t_0} - C_{t_0})^2 \\
&\ge R_{t_0}(\tilde{\varphi})
\end{aligned}$$

The equality holds iff $\tilde{C}_{t_0} = C_{t_0}$ for all $t_0 \le T$. Thus, φ is mean-self-financing. ■

These two lemmas imply that the value process of a risk-minimizing strategy has to be a martingale. Thus, $V_t(\varphi) = E[H|\mathcal{F}_t]$. This motivates the use of Kunita-Watanabe decomposition to find risk-minimizing strategies.

3.2 Kunita-Watanabe Projection

We appeal to the Kunita-Watanabe decomposition of the claim H (cf. [KS88], page 181): Given that $(X_t)_{0\le t\le T}$ is a square-integrable martingale, H can be uniquely represented as the following form:

$$H = E[H] + \int_0^T \xi_s^* dX_s + L^H \tag{7}$$

with $\xi^* \in L^2(\overline{P})$, $L^H \in L^2(P)$ has expectation 0 and is orthogonal to the space $\{\int_0^t \xi_s dX_s | \xi \in L^2(\overline{P})\}$ of stochastic integrals with respect to X.

Define

$$V_t^* = E[H|\mathcal{F}_t] = V_0^* + \int_0^t \xi_s^* dX_s + L_t^H \tag{8}$$

where $L_t^H = E[L^H|\mathcal{F}_t]$ is a right-continuous version of the square-integrable martingale with zero expectations which is orthogonal to X.

Since $(V_t)_{0\le t\le T}$ is an adapted process uniquely determined by H, ξ is uniquely determined from H using the so-called Kunita-Watanabe projection:

$$\xi^* = \frac{d\langle V, X\rangle}{d\langle X\rangle} \tag{9}$$

In fact, ξ^* forms the optimal strategy, but before proving this, we need one more useful concept:

Definition 7 *Call the following process*

$$R_t^* = E[(L^H - L_t^H)^2|\mathcal{F}_t]$$

the intrinsic risk *process of a contingent claim* H*.*

Theorem 8 *There exists unique admissible strategy* φ^* *which is risk-minimizing, namely,*

$$\varphi^* = (\xi^*, V^* - \xi^* X)$$

For this strategy, the remaining risk at any time $t \leq T$ *is given by*

$$R_t(\varphi^*) = R_t^*$$

Proof: φ^* is clearly admissible, i.e., $V_{\varphi^*}(T) = H$ P-a.s. Let φ be any admissible continuation of φ^* at time t. A little algebra reveals that

$$\begin{aligned} C_T - C_t &= V_T - V_t - \int_t^T \xi_s dX_s \\ &= V_0^* + \int_0^T \xi_s^* dX_s + L^H - V_t - \int_t^T \xi_s dX_s \\ &= \int_t^T (\xi_s^* - \xi_s) dX_s + (L^H - L_t^H) + (V_t^* - V_t) \end{aligned}$$

The fact that X and L_t^H are orthogonal, and the stochastic integral wrt X is also a martingale imply

$$\begin{aligned} E[(C_T - C_t)^2|\mathcal{F}_t] &= E[\int_t^T (\xi_s^* - \xi_s)^2 |d\langle X_s\rangle|\mathcal{F}_t] + R_t^* + (V_t^* - V_t)^2 \\ &\geq R_t^* \end{aligned}$$

This shows that φ is risk-minimizing. To show that this optimal strategy is unique, let $\tilde{\varphi} = (\tilde{\xi}, \tilde{\eta})$ be another admissible risk-minimizing strategy. The above equation implies that $\xi_t^* = \tilde{\xi}_t$ a.s. for all $0 \leq t \leq T$. Furthermore, the value process $\tilde{V}$ is also a martingale by lemma 5. But $V_T^* = \tilde{V}_T = H$ P-a.s. This implies that $V_t^* = \tilde{V}_t$ P-a.s. Hence, $\eta_t^* = \tilde{\eta}_t$ P a.s. for all $0 \leq t \leq T$. ∎

The results for attainable claims in complete market (e.g. [HK79]) can be obtained directly as a special case of the theorem above.

Corollary 9 *The following are equivalent:*
(1) The risk-minimizing admissible strategy φ^* *is self-financing*
(2) The intrinsic risk of the contingent claim H *is zero.*
(3) The contingent claim H *is attainable, i.e.,* P*-a.s.*

$$H = E[H] + \int_0^T \xi_s^* dX_s \tag{10}$$

Remark. Another important result proved by Follmer and Sonderman is that the optimal strategy φ^* does change with respect to the change of the martingale measure $P = P^*$. This is the reason why the situation becomes more delicate when we move into the general case in the next section.

4 General case: X is semimartingale

Now we consider the more general situation where X is a semimartingale, as set up in Section 2.

The question is how to identify the criterion for an "optimal" admissible strategy? While the risk-minimization concept provides a natural criterion for the case X is a martingale, it does not apply here: Recall the remaining risk equation 6, where the cost process can now be written as:

$$C_t = V_t - \int_0^t \xi_s dM_s - \int_0^t \xi_s dA_s \tag{11}$$

The problem is that we cannot control the influence of the term $\int \xi dA$ involved in the process $R(\varphi)$. Technically, there is no immediate analog to the Kunita-Watanabe decomposition that allows us to decompose a claim H into a stochastic integral of X and an orthogonal component. Intuitively, the class of variations of trading strategy is too large.

In [[]S91], Schweizer introduced the concept of *locally risk-minimizing* strategy that is "risk-minimizing" under a small perturbation. Interestingly, such a locally risk-minimizing strategy turns out to be also mean-self-financing, as defined in Section 3. Based on this criterion, there are two main approaches for finding optimal strategies. The first approach, by Schweizer in the same paper, involves deriving and solving an optimal equation for optimal strategies. This approach is addressed in this section.

The second approach, by Follmer and Schweizer [FS90], which seems to be more natural, uses a Girsanov transformaton to shift the problem back to a martingale measure $\hat{P}$, where standard techniques developed in Section 3 can be applied. Since such a measure is not unique in incomplete market (cf. ˜citeE02), the authors introduce the notion of *minimum martingale measure* that preserves the structure of the original measure P as far as possible under the constraint that X is $\hat{P}$-martingale. This approach is addressed in Section 5.

4.1 Locally Risk-minimization

Definition 10 *A strategy* $\Delta = (\delta, \epsilon)$ *is called a* small perturbation *if it satisfies the following conditions: (i)* δ *is bounded (ii)* $\int_0^T |\delta_s| d|A|_s$ *is bounded (iii)* $\delta_T = \epsilon_T = 0$

Remark. The first two conditions aim to "control" the gain caused by the "drift" A of X. The third condition ensures that if φ is an admissible strategy, then $\varphi + \Delta$ also is, and the restriction of Δ to any subinerval of $[0, T]$ is again a small perturbation.

The idea is to introduce the local variation of a strategy. For each partition $\tau = (0 = t_0 < t_1 < ... < t_N = T)$ of $[0, T]$, define the mesh to be $|\tau| = \max |t_i - t_{i-1}|$.

A sequence $(\tau_n)_{n \in N}$ of partitions is called *increasing* if $\tau_n \subset \tau_{n+1}$ for all n. It will be called 0-*convergent* if $\lim_{n\to\infty} |\tau_n| = 0$.

Define a *restriction* of Δ to a subinterval $(s, t]$ to be $\Delta|_{(s,t]} = (\delta|_{(s,t]}, \epsilon|_{[s,t)})$, where

$$\begin{aligned} \delta|_{(s,t]}(\omega, u) &= \delta_u(\omega) 1_{(s,t]}(u) \\ \epsilon|_{[s,t)}(\omega, u) &= \epsilon_u(\omega) 1_{[s,t)}(u) \end{aligned}$$

Definition 11 *Given a strategy* φ*, a small perturbation* Δ *and a partition* τ *of* $[0, T]$*, define the* risk-quotient

$$r^\tau[\varphi, \Delta](\omega, t) := \sum_{t_i \in \tau} \frac{R_{t_i}(\varphi + \Delta|_{(t_i, t_{i+1}]}) - R_{t_i}(\varphi)}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} | \mathcal{F}_{t_i}]}(\omega) 1_{(t_i, t_{i+1}]}(t) \tag{12}$$

Definition 12 *The strategy φ is called* locally risk-minimizing *if*

$$\liminf_{n\to\infty} r^{\tau_n}[\varphi,\Delta] \geq 0 \tag{13}$$

$\overline{P}$-a.e. for every small perturbation Δ and every increasing 0-convergent sequence (τ_n) of partitions of $[0,T]$.

Remark. The risk-quotient can be viewed as a measure for the total change of riskiness if φ is perturbed by Δ along a partition τ. Note that definition 12 is the infinitesimal analogue of definition 3.

Given the following additional assumptions, a locally risk-minimizing strategy is mean-self-financing.

Lemma 13 *Assume that for P-almost all ω, the measure $\overline{P}$ induced by $\langle M\rangle.(\omega)$ has the whole interval $[0,T]$ as its support. If φ is locally risk-minimizing, then it is mean-self-financing.*

Proof: The proof is also an analogue of that of Lemma 6. Construct a mean-self-financing admissible strategy $\hat{\varphi}$ such that $\hat{\xi}=\xi$. To satisfy that, for all $0\leq t\leq T$ let

$$\hat{\eta}_t = E[C_T(\varphi)|\mathcal{F}_t] + \int_0^t \xi_s dX_s - \xi_t X_t \tag{14}$$

So $\Delta = \hat{\varphi}-\varphi$ is a small perturbation. Let τ_n be the n-th dyadic partition of $[0,T]$, and denote $[d,d']$ be a subinterval $[t_j,t_{j+1}]$ of partition τ_n. We have

$$\begin{aligned} V_d(\varphi+\Delta|_{(d,d']}) &= V_d(\hat{\varphi}) \\ V_T(\varphi+\Delta|_{(d,d']}) &= V_T(\hat{\varphi}) \end{aligned}$$

So $C_T(\varphi+\Delta|_{(d,d']}) - C_d(\varphi+\Delta|_{(d,d']}) = C_T(\hat{\varphi}) - C_d(\hat{\varphi})$

So $R_d(\varphi+\Delta|_{(d,d']}) = R_d(\hat{\varphi})$. Now

$$\begin{aligned} R_d(\hat{\varphi}) &= E[(C_T(\hat{\varphi})-C_d(\hat{\varphi}))^2|\mathcal{F}_d] \\ &= E[(C_T(\varphi)-C_d(\varphi)+C_d(\varphi)-C_d(\hat{\varphi}))^2|\mathcal{F}_d] \\ &= R_d(\varphi)+2(C_d(\varphi)-C_d(\hat{\varphi}))E[(C_T(\varphi)-C_d(\varphi))|\mathcal{F}_d]+(C_d(\varphi)-C_d(\hat{\varphi}))^2 \\ &= R_d(\varphi)-(C_d(\varphi)-C_d(\hat{\varphi}))^2 \end{aligned}$$

Hence

$$\begin{aligned} R_d(\varphi+\Delta|_{(d,d']})-R_d(\varphi) &= R_d(\hat{\varphi})-R_d(\varphi) \\ &= -(C_d(\varphi)-C_d(\hat{\varphi}))^2 \\ &= -(C_d(\varphi)-E[C_T(\varphi)|\mathcal{F}_d])^2 \end{aligned}$$

Thus

$$r^{\tau_n}[\varphi,\Delta] = -\sum_{d\in\tau_n} \frac{(C_d(\varphi)-E[C_T(\varphi)|\mathcal{F}_d])^2}{E[\langle M\rangle_{d'}-\langle M\rangle_d|\mathcal{F}_d]}.1_{(d,d']} \tag{15}$$

Suppose that for some dyadic rational d_0, there is a set B with $P(B)>0$ such that for all $\omega\in B$,

$$C_{d_0}(\varphi) \neq E[C_T(\varphi)|\mathcal{F}_{d_0}] \tag{16}$$

By the right-continuity of $C.(\varphi)$ and $E[C_T(\varphi)|\mathcal{F}.]$ this imply that for each $\omega\in B$ equation 16 holds true for all $d\in[d_0,d_0+\gamma(\omega)]$ for some constant $\gamma(\omega)$. But then equation 15 contradicts the definition 12.

Hence, $P(B) = 0$ for any dyadic rational d_0, which implies $P(B) = 0$ for any $0 \leq t \leq T$, again by the right-continuity of $C.(\varphi)$ and $E[C_T(\varphi)|\mathcal{F}_.]$. ■

This lemma and its proof tell us that, we can find a locally risk-minimizing strategy by varying only the ξ component, because η can be uniquely determined to ensure that $C(\varphi)$ is a martingle. Several additional assumptions are needed to "control" the influence of the drift component A of X.

Assumption. (i) A is continuous
(ii)A is absolutely continous with respect to $\langle M\rangle$ with a density α satisfying $E_M[|\alpha|.\log^+\|\alpha|] < \infty$.
(iii)X is continous at T P-a.s.

Technically, the risk-quotient $r^\tau[\varphi, \Delta]$ can be decomposed into 2 terms, one of which depends on only ξ, δ, and another depending on ϵ and drift A, which is negligible by the assumption. The local risk-minimization of φ is reduced to that of the ξ under a local perturbation. Along this line, Schweider [S90] was able to obtain a martingale-theoretic characterization of locally risk-minimizing as follows:

Proposition 14 *Let φ be an H-admissible strategy. Under the assumption above, φ is locally risk-minimizing if and only if φ is mean-self-financing and the martingale $C(\varphi)$ is orthogonal to M under P.*

This important proposition makes it possible to derive the the necessary and sufficient condition for the existence of optimal trading strategies.

4.2 Necessary and sufficient conditions

Definition 15 *An admissible strategy φ is called* optimal *if the associated cost process $C(\varphi)$ is a martingale which is orthogonal to M under P.*

Theorem 16 *The existence of an optimal strategy is equivalent to a decomposition of contingent claim H:*

$$H = H_0 + \int_0^T \xi_s dX_s + L^H \tag{17}$$

where $H_0 = E[H|\mathcal{F}_0] \in L^2(\Omega, \mathcal{F}_0, P)$, $\xi \in L^2(\overline{P})$, and $L^H_t = E[L^H|\mathcal{F}_t]$ is a square-integrable (right-continuous) martingale of zero expectation and is orthogonal to M under P.

Proof: $(\Rightarrow)$: Given the representation 17 of H, it is easy to see that the optimal strategy is $\varphi = (\xi, V - \xi X)$, where

$$V_t = H_0 + \int_0^t \xi_s dX_s + L^H_t \tag{18}$$

The cost process becomes $C_t = H_0 + L^H_t$ satisfying the optimality criterion.

$(\Leftarrow)$: An optimal strategy φ means $C_t = E[C_T|\mathcal{F}_t]$ is a martingale orthogonal to M, which leads to the decomposition

$$\begin{aligned} H = V_T &= C_T + \int_0^T \xi_s dX_s \\ &= H_0 + \int_0^T \xi_s dX_s + (C_T - C_0) \end{aligned}$$

■

Remark. Note the similarity between equations 17 and 18 with 7 and 8 in the previous section. The important difference is that here X is no longer a martingale. As a result, neither is V. This implies that we can no longer apply the Kunita-Watanabe projection to find ξ and η.

4.3 Deriving the optimal equation

Finding the optimal trading strategy amounts to finding the decomposition 17 for claim H. An approach taken in [S91] is to attack this equation directly.

Applying the Kunita-Watanabe decomposition to H and the square-integrable martingale M:

$$H = N_0 + \int_0^T \mu_s dM_s + N^H \tag{19}$$

where $N_t = E[N^H|\mathcal{F}_t]$ is a martingale of zero expectation and is orthogonal to M. Applying the Kunita-Watanabe decomposition to $\int \xi dA$ to have:

$$\int_0^T \xi_s dA_s = N_0^\xi + \int_0^T \mu_s^\xi dM_s + N^\xi \tag{20}$$

where $N_t^\xi = E[N^\xi|\mathcal{F}_t]$ is a martingale of zero expectation and is orthogonal to M. Combining equation 17 and 20 gives

$$H = (H_0 + N_0^\xi) + \int_0^T (\xi_s + \mu_s^\xi) dM_s + (L^H + N^\xi) \tag{21}$$

Since the Kunita-Watanabe decomposition is unique, from this and equation 19, we have for any $0 \leq s \leq T$, P-almost surely

$$\xi_s + \mu_s^\xi = \mu_s \tag{22}$$

One can now focus on solving this *optimality equation*, which is the approach taken in [S91].

5 Minimal martigale measure approach

This section presents a more natural approach introduced in [FS90]. The basic idea is to use a Girsanov transformation to shift the problem back into a martingale measure from which the optimal trading strategy can be computed in a manner similar to techniques in Section 3. In incomplete markets, there may be more than one equivalent martingale measure. The martingale measure that we consider has to preserve the structure of the original P as much as possible. Now we will make all this precise.

Definition 17 *A martingale measure $\hat{P} \approx P$ is called* mimimal *if*
(i) $\hat{P} = P$ on $\mathcal{F}_0$
(ii) If L is a square-integrable P-martingale orthorgonal to M under P,(i.e. $\langle L, M\rangle = 0$ P-a.s.) then L is a $\hat{P}$-martingale.

Note that an equivalent martingale measure P^* is uniquely determined by the right-continuous square-integrable P-martingale G^* with

$$G_t^* = E[\frac{dP^*}{dP}|\mathcal{F}_t] \tag{23}$$

The Doob-Myer decomposition of X under P is $X = X_0 + M + A$. Hence, the Doob-Myer decomposition of M under P^* is $M = -X_0 + X + (-A)$. The theory of Girsanov transformation shows that the predictable process $-A$ of bounded variation can be computed in terms of G^*:

$$-A_t = \int_0^t \frac{1}{G_{s-}} d\langle M, G^*\rangle_s \ (0 \leq t \leq T) \tag{24}$$

Since $\langle M, G^*\rangle \ \ll \langle M\rangle = \langle X\rangle$, the process A must be absolutely continuous wrt the variance process $\langle X\rangle$ of X. Hence, A can be written as

$$A_t = \int_0^t \alpha_s d\langle X\rangle_s \ (0 \leq t \leq T) \tag{25}$$

for some predictable process $\alpha = (\alpha_t)_{0\leq t\leq T}$.

This sets up the following theorem on the existence and uniqueness of the minimal martingale measure.

Theorem 18 *$\hat{P}$ exists if and only if*

$$\hat{G}_t = \exp(-\int_0^t \alpha_s dM_s - \frac{1}{2}\int_0^t \alpha_s^2 d\langle X\rangle_s) \ (0 \leq t \leq T) \tag{26}$$

is a square-integrable P-martingale. Under this condition, $\hat{P}$ is uniquely determined by

$$\hat{G}_T = \frac{d\hat{P}}{dP} \tag{27}$$

Remark. This result generalizes the Girsanov change of measure for Brownian motion with drift [E02,KS88].
Proof: (Sketch). $(\Rightarrow)$: Assume that the equivalent martingale measure P^* is minimal. Let G^* be the square-integrable martingale associated with $P^* \approx P$. According to the Kunita-Watanabe decomposition, we have

$$G_t^* = G_0^* + \int_0^t \beta_s dM_s + L_t \ \ (0 \leq t \leq T) \tag{28}$$

where L is a square-integrable martingale under P orthogonal to M and β is a predictable process with

$$E[\int_0^T \beta_s^2 d\langle M\rangle] < \infty \tag{29}$$

We have

$$A_t = \int_0^t \frac{1}{G_{s-}^*} d\langle G^*, M\rangle_s \quad = \quad \int_0^t \frac{1}{G_{s-}^*} \beta_s d\langle X\rangle_s$$

This implies

$$\alpha = -\frac{\beta}{G_-^*} \tag{30}$$

Since $G^* \geq 0$ P-a.s due to $P^* \approx P$ and since $\langle M\rangle = \langle X\rangle$, the condition 29 implies

$$\int_0^T \alpha_s^2 d\langle X\rangle_s < \infty \ P\text{-}a.s. \tag{31}$$

Since P^* is assumed to be minimal measure, $G_0^* = E[dP^*/dP|\mathcal{F}_0] = 1$; and L is a martingale under P^*. This implies $\langle L, G^*\rangle = 0$, and so we get

$$\langle L\rangle = \langle L, G^*\rangle = 0 \tag{32}$$

hence $L \equiv 0$. Therefore, G^* solves the stochastic equation

$$G^*_t = 1 + \int_0^t G^*_{s-}(-\alpha_s) dM_s \tag{33}$$

which has the solution $P^* = \hat{P}$ given in 26.

($\Leftarrow$): Assume that the the process $\hat{G}$ given in 26 is a square-integrable martingale. We need to show that the associated martingale measure $\hat{P}$ is indeed minimal. Let L be a square-integrable P-martingale and $\langle L, M\rangle = 0$. We need to show that L is also a $\hat{P}$-martingale.

Since L solves the stochastic equation 33, we get $\langle L, \hat{G}\rangle = 0$, and so L is a local martingale under $\hat{P}$. To see why L is actually a $\hat{P}$-martingale, note that L and G^T is a square-integrable P-martingale. Applying maximal inequality, we have

$$\begin{aligned} \hat{E} \sup_{0\le t\le T} |L_t| &= E \sup_{0\le t\le T} |L_t| \hat{G}_T \\ &\le (E \sup_{0\le t\le T} L_t^2 E\hat{G}_T^2)^{1/2} \\ &\le (4EL_T^2 E\hat{G}_T^2)^{1/2} < \infty \end{aligned}$$

Hence, the local martingale L is in fact a martingale under $\hat{P}$. ■

The most important characteristic of the minimal martingale measure that is essential to our use is:

Theorem 19 *$\hat{P}$ preserves orthogonality: If L is a square-integrable martingale in P orthogonal to M, then L is orthogonal to X under $\hat{P}$.*

Proof: (cf. [FS90]). ■

Loosely speaking, the theorem above says that the minimal martingale measure $\hat{P}$ preserves the martingale property of P as far as possible. In fact, this minimal departure from a given measure P can be expressed in terms of the *relative entropy* (or the Kullback-Leibler divergence)

$$H(Q|P) := \int \log \frac{dQ}{dP} dQ \ \ if\ Q \ll P \tag{34}$$

and 0 otherwise.

Theorem 20 *If $P^* \approx P$ is an equivalent martingale measure,*

$$H(P^*|P) \ge \frac{1}{2} E^*[\int_0^T \alpha_s^2 d\langle X\rangle_s] \tag{35}$$

Equality holds when $P^* = \hat{P}$.

Proof: (cf. [FS89]). ■

We have developed enough tools for finding the optimal strategy. Indeed, given the decomposition 17 of the contingent claim H, to be rewritten here:

$$H = H_0 + \int_0^T \xi_s dX_s + L^H \tag{36}$$

where L_t^H is a square-integrable P-martingale orthogonal to M. But the definition of $\hat{P}$ and theorem 19 implies that L_t^H is a $\hat{P}$-martingale orthogonal to X. Hence, the equation 36 becomes a Kunita-Watanabe decomposition of H under the minimal martingale measure $\hat{P}$. The rest is a repetition of what we have done in Section 3. The right-continuous version of the martingale $V_t = \hat{E}[H|\mathcal{F}_t]$ has the form:

$$V_t = H_0 + \int_0^t \xi_s dX_s + L_t^H \tag{37}$$

Thus we have proved the main theorem:

Theorem 21 *The optimal strategy (ξ, η) is uniquely determined by $\hat{P}$, in which:*

$$\xi = \frac{d\langle V, X\rangle}{d\langle X\rangle} \tag{38}$$

6 Incompleteness due to incomplete information

In this section, we consider a situation in which the incompleteness of the market comes from the lack of information. More precisely, let the information accessible to us be described by the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$. Suppose that the claim H is attainable wrt a larger (right-continuous) filtration $(\tilde{\mathcal{F}}_t)_{0\leq t\leq T}$:

$$\begin{aligned} \mathcal{F}_t &\subset \tilde{\mathcal{F}}_t \subset \mathcal{F}\ (0 \leq t \leq T) \\ \mathcal{F}_T &= \tilde{\mathcal{F}}_T \end{aligned}$$

Furthermore, assume that the Doob-Myer decomposition of X wrt $(\mathcal{F}_t)$ is still valid for $(\tilde{\mathcal{F}}_t)$, i.e. M is a $P-$ martingale wrt $(\tilde{\mathcal{F}}_t)$ although it is adapted to the smaller filtration $(\mathcal{F}_t)$.

By our assumption, H can be written as:

$$H = \tilde{H}_0 + \int_0^T \tilde{\xi}_s dX_s \tag{39}$$

where $\tilde{H}_0$ is $\mathcal{F}_0$-measurable, $\tilde{\xi}$ is predictable wrt $(\tilde{\mathcal{F}}_t)$.

We also assume that the $(\tilde{\mathcal{F}}_t)$-semimartingale $\tilde{H}_0 + \int_0^t \tilde{\xi}_s dX_s$ are "square-integrable", i.e.,

$$E[\tilde{H}_0^2 + \int_0^T (\tilde{\xi}_s^H)^2 d\langle X\rangle_s + (\int_0^T |\tilde{\xi}_s^H| d|A|_s)^2] < \infty \tag{40}$$

Recall the probability space $(\Omega \times [0,T], \mathcal{P}, \overline{P})$ define in Section 2. Define $\tilde{\mathcal{P}}$ to be the σ-field of predictable sets on $(\Omega \times [0,T])$ associated with the filtration $(\tilde{\mathcal{F}}_t)$.

The following theorem shows that the useful decomposition of H that allows the existence (and uniqueness) of optimal trading strategies indeed exists by projecting the components of the representation 39 back to the original probability space.

Theorem 22 *Assume that H satisfies 39 and 40. Then H admits the representation*

$$H = H_0 + \int_0^T \xi_s dX_s + L^H \tag{41}$$

where

$$\begin{aligned} H_0 &:= E[\tilde{H}_0|\mathcal{F}_0] \\ \xi &:= \overline{E}[\tilde{\xi}|\mathcal{P}] \\ L^H &:= \tilde{H}_0 - H_0 + \int_0^T (\tilde{\xi}_s - \xi_s)dX_s \quad \in L^2(\Omega, \mathcal{F}_T, P) \end{aligned}$$

in which $(L^H_t = E[L^H|\mathcal{F}_t])_{0\le t\le T}$ *is a square-integrable martingale orthogonal to* M*.*

Proof: (Sketch). First of all, we need to show that all components in the decomposition 22 are square-integrable. Since $\int_0^T \tilde{\xi}_s dX_s$ is square-integrable by the integrability condition 40, what remains is to show the square-integrability for $\int_0^T \xi_s dX_s = \int_0^T \xi_s dM_s + \int_0^T \xi_s dA_s$. Since $\xi \in L^2(\Omega\times[0,T], \mathcal{P}, \overline{P})$, by the maximal inequality we get $\int_0^T \xi_s dM_s \in L^2(\Omega, \mathcal{F}_T, P)$. As for $\int_0^T \xi_s dA_s$, applying the representation 25 and using the maximal inequality gives the square-integrability result.

Now, we need to show that L_H is orthogonal to M. It is sufficient to show that for any bounded $\mathcal{P}$-measurable process $\mu = (\mu_t)_{0\le t\le T}$ the following holds:

$$\begin{aligned} & E[(\int_0^T (\tilde{\xi}_s - \xi_s)dX_s).(\int_0^T \mu_s dM_s)] = 0 \\ \Leftrightarrow \quad & E[(\int_0^T \tilde{\xi}_s dX_s).(\int_0^T \mu_s dM_s)] = E[(\int_0^T \xi_s dX_s).(\int_0^T \mu_s dM_s)] \end{aligned}$$

But the left hand side can be decomposed into two components (using the Ito-like isometry)

$$E[(\int_0^T \tilde{\xi}_s dM_s).(\int_0^T \mu_s dM_s)] = E[\int_0^T \tilde{\xi}_s \mu_s d\langle X\rangle_s]$$

and

$$E[(\int_0^T \tilde{\xi}_s dA_s).(\int_0^T \mu_s dM_s)] = E[\int_0^T \tilde{\xi}_s.(\int_0^s \mu_u dM_u).\alpha_s d\langle X\rangle]$$

Now, in both parts, $\tilde{\xi}$ can be replaced by ξ, which gives what we need. ■

Remark. The decomposition 22 implies the unique existence of an optimal trading strategy by Theorem 16.

Now, with the decomposition 22 in hands, we are back to the familiar territory, because this decomposition allows us to compute the optimal (ξ, η) from the minimal martingale $\hat{P}$. It turns out that in our context, ξ can be computed directly from the $\tilde{\xi}$ by taking the conditional expectation wrt $\hat{P}$. Let the measure $\overline{\hat{P}}$ for $\hat{P}$ be the counterpart of $\overline{P}$ for P.

Theorem 23 *The optimal strategy is uniquely given by*

$$\begin{aligned} \xi &= \overline{\hat{E}}[\tilde{\xi}|\mathcal{P}] \\ \eta &= V - \xi X \end{aligned}$$

where $V_t = \hat{E}[H|\mathcal{F}_t]$

Proof: (Sketch). It is sufficient to consider $\tilde{\xi} \ge 0$ (otherwise we can decompose it into the difference of two non-negative terms). We need to show $\overline{\hat{E}}[\tilde{\xi}|\mathcal{P}] = \xi$, where

$$\xi = \overline{E}[\tilde{\xi}|\mathcal{P}] \tag{42}$$

It is equivalent to showing

$$\hat{E}[\int_0^T \tilde{\xi}_s \vartheta_s d\langle X\rangle_s] = \hat{E}[\int_0^T \xi_s \vartheta_s d\langle X\rangle_s] \quad (43)$$

for any non-negative $\mathcal{P}$-measurable process ϑ. By definition of $\hat{P}$ (Theorem 18), the left hand side equals

$$\begin{aligned} LHS &= E[\hat{G}_T \int_0^T \tilde{\xi}_s \vartheta_s d\langle X\rangle_s] \\ &= E[\int_0^T \hat{G}_s \tilde{\xi}_s \vartheta_s d\langle X\rangle_s] \text{ (by predictable projection)} \\ &= E[\int_0^T \hat{G}_s \xi_s \vartheta_s d\langle X\rangle_s] \text{ by definition of } \xi) \\ &= E[\hat{G}_T \int_0^T \xi_s \vartheta_s d\langle X\rangle_s] \\ &= RHS \end{aligned}$$

■

7 Concluding remark

This report surveys a range of important ideas for hedging contingent claims in incomplete markets. The main machinery is that of the powerful martingale theory. Especially notable is the effective exploitation of several techniques, such as Kunita-Watanabe projection and Girsanov change of measure.

The assumptions and results reported herein are general enough to capture a wide range of application. In [FS86] the authors considered an example of stock price that follows a two-sided jump process and computed explicitly the intrinsic risk and the risk-minimizing strategy for a call option under the setting $P \equiv P^*$ (Section 3). In [S91], the authors studied in detail the case in which the incompleteness comes from a random fluctuation in the variance of the stock price, which is a geometric Brownian motion, and computed the optimal strategy using the techniques surveyed in Sections 5 and 6.

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