On the association of sum- and max-stable processes

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ARTICLE INFO	ABSTRACT
	We address the notion of association of sum- and max-stable processes from the perspective of linear and max-linear isometries. We establish the appealing result that these two classes of isometries can be identified on a proper space (the extended positive ratio space). This yields a natural way to associate to any max-stable process a sum-stable process. By using this association, we establish connections between structural and classification results for sum- and max-stable processes. © 2009 Elsevier B.V. All rights reserved.

1. Introduction

Sum- and max-stable processes are two classes of stochastic processes, which have been investigated for a long time. Many results are available about their representations, stochastic properties, and structure (see e.g. Samorodnitsky and Taqqu, 1994, Rosiński, 1995, 2000, Samorodnitsky, 2005, Pipiras and Taqqu, 2002, 2004 and Roy and Samorodnitsky, 2008, as well as, e.g. de Haan, 1984, de Haan and Pickands, 1986, Davis and Resnick, 1993, Stoev and Taqqu, 2006, de Haan and Ferreira, 2006, and Resnick, 2007).

The deep connection between heavy-tailed sum- and max-stable processes has long been suspected. As observed, for example, in Davis and Resnick (1989) the moving maxima and the moving averages are statistically indistinguishable in the extremes. Also, the maxima of independent copies of a sum-stable process (appropriately rescaled) converge in distribution to a max-stable process and the two processes have very similar spectral representations (see e.g. Theorem 5.1 in Stoev and Taqqu (2006)). In Stoev (2008), the ergodic properties of max-stable processes were characterized by borrowing ideas and drawing parallels to existing work in the sum-stable domain. Recently Kabluchko (2009) codified the notion of association between sum- and max-stable processes. This was done by essentially identifying the spectral measures of the finite-dimensional distributions of the two processes. In this paper we provide another, equivalent, definition of association between sum- and max-stable processes, which is based on relating their spectral functions. The combination of two perspectives provides a more clear picture about the fundamental connections between the two classes of processes.

In the rest of this section, we briefly introduce the main results of this paper. We will only consider infinite variance *symmetric α-stable* (SαS, $\alpha \in (0, 2)$) sum-stable processes and *α-Fréchet* max-stable processes. Recall that an infinite variance SαS variable X has characteristic function $\varphi_X(t) = \mathbb{E}\exp\{-itX\} = \exp\{-\sigma^\alpha |t|^\alpha\}$, $\forall t \in \mathbb{R}$, where $\alpha \in (0, 2)$. On the other hand, Y has an α-Fréchet distribution if $F_Y(y) = \mathbb{P}(Y \leq y) = \exp\{-\sigma^\alpha y^{-\alpha}\}$, $\forall y \in (0, \infty)$, where now α is in $(0, \infty)$. The σ's in both cases are positive parameters referred to as *scale coefficients*. Observe that the form of the characteristic function φ_X of an SαS variable is preserved under multiplication, which reflects the sum-stability of this class of distributions. Similarly, the form of the distribution function F_Y is preserved under multiplication, which corresponds to stability with respect to taking independent maxima.

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More similarities can be observed on the level of processes. Recall that $X = \{X_t\}_{t \in T}$ is an SαS stochastic process if all its finite linear combinations $\sum_{i=1}^n a_i X_{t_i}$, $a_i \in \mathbb{R}$, $t_i \in T$, are SαS. These processes have convenient integral (or spectral) representations:

$$\{X_t\}_{t \in T} \stackrel{\mathrm{d}}{=} \left\{ \int_S f_t(s) M_{\alpha,+}(\mathrm{d}s) \right\}_{t \in T}. \tag{1.1}$$

Here $\{f_t\}_{t \in T} \subset L^\alpha(S, \mu)$, '$\int$' stands for the *stable integral* and $M_{\alpha,+}$ is an *SαS random measure* on measure space (S, μ) with control measure μ (see Chapters 3 and 13 in Samorodnitsky and Taqqu, 1994).

On the other hand, $Y = \{Y_t\}_{t \in T}$ is an α-Fréchet max-stable process if all its *positive* max-linear combinations $\bigvee_{i=1}^n a_i Y_{t_i}$, $a_i \geq 0$, $t_i \in T$ are α-Fréchet. Such processes have extremal integral representations of the form

$$\{Y_t\}_{t \in T} \stackrel{\mathrm{d}}{=} \left\{ \int_S^e f_t(s) M_{\alpha,\vee}(\mathrm{d}s) \right\}_{t \in T}. \tag{1.2}$$

Here $\{f_t\}_{t \in T} \subset L_+^\alpha(S, \mu) := \{f \in L^\alpha(S, \mu) : f \geq 0\}$, '$\int^e$' stands for the *extremal integral* and $M_{\alpha,\vee}$ is an *α-Fréchet random sup-measure* with control measure μ (see e.g. Stoev and Taqqu, 2006 and de Haan, 1984 for an alternative treatment). The f_t's in (1.1) and (1.2) are called the *spectral functions* of the sum- or max-stable processes, respectively. In this paper, we will assume (S, μ) to be a standard Lebesgue space (see Appendix A in Pipiras and Taqqu, 2004) and let T denote an arbitrary index set, which may be sometimes equipped with a measure λ. Two common settings are $(S, \mu) = ([0, 1], \text{Leb})$, $T = \mathbb{Z}$ with λ being the counting measure and $T = \mathbb{R}$ with λ being the Lebesgue measure. A brief review on stable and extremal integrals is given in Table 1 in Section 2.

The representation (1.1) implies that

$$\mathbb{E} \exp\left\{ -\mathrm{i} \sum_{j=1}^n a_j X_{t_j} \right\} = \exp\left\{ -\int_S \left| \sum_{j=1}^n a_j f_{t_j}(s) \right|^\alpha \mu(\mathrm{d}s) \right\}, \quad a_j \in \mathbb{R}, \, t_j \in T, \tag{1.3}$$

which determines the finite-dimensional distributions (f.d.d.) of the SαS process $\{X_t\}_{t \in T}$. The f.d.d. of the α-Fréchet process $\{Y_t\}_{t \in T}$ in (1.2), on the other hand, are expressed as:

$$\mathbb{P}(Y_{t_1} \leq a_1, \ldots, Y_{t_n} \leq a_n) = \exp\left\{ -\int_S \left(\bigvee_{j=1}^n f_{t_j}(s)/a_j \right)^\alpha \mu(\mathrm{d}s) \right\}, \quad a_j \geq 0, \, t_j \in T. \tag{1.4}$$

Note that the r.h.s. of (1.3) and (1.4) are similar. Indeed, they both involve exponentials of either linear ($\sum$) or max-linear ($\bigvee$) combinations of spectral functions. The characterizations (1.3) and (1.4) and their close connections play an important role throughout this paper. In particular, the key result behind our notion of association is the following theorem.

Theorem 1.1. *Consider two arbitrary collections of functions $f_1^{(i)}, \ldots, f_n^{(i)} \in L_+^\alpha(S_i, \mu_i)$, $i = 1, 2$, $0 < \alpha < 2$. Then,*

$$\left\| \sum_{j=1}^n a_j f_j^{(1)} \right\|_{L^\alpha(S_1, \mu_1)} = \left\| \sum_{j=1}^n a_j f_j^{(2)} \right\|_{L^\alpha(S_2, \mu_2)}, \quad \text{for all } a_j \in \mathbb{R}, \tag{1.5}$$

if and only if

$$\left\| \bigvee_{j=1}^n a_j f_j^{(1)} \right\|_{L_+^\alpha(S_1, \mu_1)} = \left\| \bigvee_{j=1}^n a_j f_j^{(2)} \right\|_{L_+^\alpha(S_2, \mu_2)}, \quad \text{for all } a_j \geq 0. \tag{1.6}$$

The proof of this result is given in Section 3. We now briefly discuss its role. It is well known that the spectral representations in (1.1) and (1.2) are not unique. Let $\{g_t^{(i)}\}_{t \in T} \subset L_+^\alpha(S_i, \mu_i)$, $i = 1, 2$ be two spectral representations of an SαS process $\{X_t\}_{t \in T}$. Relation (1.3) shows that (1.5) holds with $g_{t_j}^{(i)}$ replaced by $f_j^{(i)}$, $i = 1, 2$. Relation (1.4) and Theorem 1.1 then imply that $\{g_t^{(i)}\}_{t \in T}$, $i = 1, 2$ are two spectral representations of the same α-Fréchet process $\{Y_t\}_{t \in T}$. Conversely, two equivalent spectral representations of an α-Fréchet process $\{Y_t\}_{t \in T}$ yield also equivalent representations of an SαS process $\{X_t\}_{t \in T}$. Therefore, we define the association as follows:

Definition 1.1 (*Associated SαS and α-Fréchet Processes*). We say that an SαS process $\{X_t\}_{t \in T}$ and an α-Fréchet process $\{Y_t\}_{t \in T}$ are *associated*, if there exist $\{f_t\}_{t \in T} \subset L_+^\alpha(S, \mu)$ such that:

$$\{X_t\}_{t \in T} \stackrel{\mathrm{d}}{=} \left\{ \int_S f_t \mathrm{d}M_{\alpha,+} \right\}_{t \in T} \quad \text{and} \quad \{Y_t\}_{t \in T} \stackrel{\mathrm{d}}{=} \left\{ \int_S^e f_t \mathrm{d}M_{\alpha,\vee} \right\}_{t \in T}.$$

In this case, we say $\{X_t\}_{t \in T}$ and $\{Y_t\}_{t \in T}$ are associated by $\{f_t\}_{t \in T}$.

Table 1
Basic properties of stable and extremal stochastic integrals.

SαS, $\alpha \in (0,2), f \in L^\alpha(S,\mu)$	α-Fréchet, $\alpha \in (0,\infty), f \in L^\alpha_+(S,\mu)$
Stochastic integral:	*Extremal integral:*
$Z = \int_S f(s)M_{\alpha,+}(\mathrm{d}s)$ is SαS	$Z = {}^e\!\int_S f(s)M_{\alpha,\vee}(\mathrm{d}s)$ is α-Fréchet
Scale coefficient:	*Scale coefficient:*
$\|Z\|_\alpha = (\int_S |f|^\alpha \mathrm{d}\mu)^{1/\alpha}$	$\|Z\|_\alpha = (\int_S f^\alpha \mathrm{d}\mu)^{1/\alpha}$
Independently scattered:	*Independently scattered:*
$\int_S f_1 \mathrm{d}M_{\alpha,+} \perp\!\!\!\perp \int_S f_2 \mathrm{d}M_{\alpha,+} \Leftrightarrow f_1 f_2 = 0$	${}^e\!\int_S f_1 \mathrm{d}M_{\alpha,\vee} \perp\!\!\!\perp {}^e\!\int_S f_2 \mathrm{d}M_{\alpha,\vee} \Leftrightarrow f_1 f_2 = 0$
Linearity: $\forall a_i \in \mathbb{R}, f_i \in L^\alpha(S,\mu)$	*Max-linearity:* $\forall a_i \in \mathbb{R}_+, f_i \in L^\alpha_+(S,\mu)$
$\int_S (a_1 f_1 + a_2 f_2)\mathrm{d}M_{\alpha,+} = a_1 \int_S f_1 \mathrm{d}M_{\alpha,+} + a_2 \int_S f_2 \mathrm{d}M_{\alpha,+}$	${}^e\!\int_S (a_1 f_1 \vee a_2 f_2)\mathrm{d}M_{\alpha,\vee} = a_1 {}^e\!\int_S f_1 \mathrm{d}M_{\alpha,\vee} \vee a_2 {}^e\!\int_S f_2 \mathrm{d}M_{\alpha,\vee}$

Theorem 1.1 ensures the consistency of this definition (see Theorem 4.1) and it also shows that our notion of association is not merely formal. For example, stationary or self-similar max-stable processes are associated with stationary or self-similar sum-stable ones, respectively (see Corollary 4.1).

We will also see, however, that there are SαS processes that cannot be associated to any α-Fréchet processes (see Theorem 4.2). In particular, we provide a practical characterization of the *max-associable* SαS processes $\{X_t\}_{t \in T}$ with stationary increments characterized by dissipative flow, indexed by $T = \mathbb{R}$ or $T = \mathbb{Z}$ (see Proposition 4.1).

The paper is organized as follows. In Section 2, some preliminaries are provided. In Section 3, we prove Theorem 1.1. In Section 4, we establish the association of SαS and α-Fréchet processes and give examples of both max-associable and non-max-associable SαS processes. In Section 5, we show how the association can serve as a tool to translate available structural results for SαS processes to α-Fréchet processes, and vice versa. We conclude with a comparison between our approach and that of Kabluchko (2009) in Section 6.

2. Preliminaries

The basic properties of the stable and extremal stochastic integrals involved in Representations (1.1) and (1.2) are summarized in Table 1. For more details, see e.g. Samorodnitsky and Taqqu (1994) and Stoev and Taqqu (2006).

In the rest of this section, we draw a connection between the linear isometries and max-linear isometries, which play important roles in relating two representations of a given SαS or an α-Fréchet process, respectively. The notion of a linear isometry is well known. To define a *max-linear isometry*, we say that a subset $\mathcal{F} \subset L^\alpha_+(S,\mu)$ is a *max-linear space* if for all $n \in \mathbb{N}, f_i \in \mathcal{F}, a_i > 0, \bigvee_{i=1}^n a_i f_i \in \mathcal{F}$ and if $\mathcal{F}$ is closed w.r.t. the metric $\rho_{\mu,\alpha}$ defined by $\rho_{\mu,\alpha}(f,g) = \int_S |f^\alpha - g^\alpha|\mathrm{d}\mu$.

Definition 2.1 (*Max-linear Isometry*). Let $\alpha > 0$ and consider two measure spaces (S_1, μ_1) and (S_2, μ_2) with positive and σ-finite measures μ_1 and μ_2. Let $\mathcal{F}_1 \subset L^\alpha_+(S_1, \mu_1)$ be a max-linear space. A mapping $U : \mathcal{F}_1 \to L^\alpha_+(S_2, \mu_2)$, is said to be a max-linear isometry, if: (i) for all $f_1, f_2 \in \mathcal{F}_1$ and $a_1, a_2 \geq 0$, $U(a_1 f_1 \vee a_2 f_1) = a_1(Uf_1) \vee a_2(Uf_2)$, μ_2-a.e. and (ii) for all $f \in \mathcal{F}_1$, $\|Uf\|_{L^\alpha_+(S_2, \mu_2)} = \|f\|_{L^\alpha_+(S_1, \mu_1)}$.

A linear (max-linear resp.) isometry may be defined only on a small linear (max-linear resp.) subspace of $L^\alpha(S,\mu)$ ($L^\alpha_+(S,\mu)$ resp.). However, this linear (max-linear resp.) isometry can be extended uniquely to the *extended ratio space* (*extended positive ratio space* resp.), which will turn out to be closed w.r.t. both linear and max-linear combinations.

Definition 2.2 (*Extended Ratio Spaces*). Let F be a collection of functions in $L^\alpha(S,\mu)$.

(i) The *ratio σ-field* of F, written $\rho(F) := \sigma(\{f_1/f_2, f_1, f_2 \in F\})$, is defined as the σ-field generated by ratio of functions in F, with the conventions $\pm 1/0 = \pm\infty$ and $0/0 = 0$.
(ii) The *extended ratio space* of F, written $\mathcal{R}_e(F)$, is defined as:

$$\mathcal{R}_e(F) := \{rf : rf \in L^\alpha(S,\mu), r \sim \rho(F), f \in F\}. \tag{2.1}$$

Similarly, we define *extended positive ratio space*:

$$\mathcal{R}_{e,+}(F) := \{rf : rf \in L^\alpha_+(S,\mu), r \sim \rho(F), r \geq 0, f \in F\}. \tag{2.2}$$

The following result is due to Hardin (1981) and Wang and Stoev (2009).

Theorem 2.1. *Let $\mathcal{F}$ be a linear (max-linear resp.) subspace of $L^\alpha(S_1, \mu_1)$ with $0 < \alpha < 2$ ($L^\alpha_+(S_1, \mu_1)$ with $0 < \alpha < \infty$ resp.). If U is a linear (max-linear resp.) isometry from $\mathcal{F}$ to $U(\mathcal{F})$, then U can be uniquely extended to a linear (max-linear resp.) isometry $\overline{U} : \mathcal{R}_e(\mathcal{F}) \to \mathcal{R}_e(U(\mathcal{F}))$ ($\overline{U} : \mathcal{R}_{e,+}(\mathcal{F}) \to \mathcal{R}_{e,+}(U(\mathcal{F}))$ resp.), with the form*

$$\overline{U}(rf) = \overline{T}(r)U(f), \tag{2.3}$$

for all $rf \in \mathcal{R}_e(\mathcal{F})$ in (2.1) ($rf \in \mathcal{R}_{e,+}(\mathcal{F})$ as in (2.2) resp.). Here $\overline{T}$ is the mapping from $L^\alpha(S_1, \rho(\mathcal{F}), \mu_1)$ to $L^\alpha(S_1, \rho(U(\mathcal{F})), \mu_2)$, induced by a regular set isomorphism T from $\rho(\mathcal{F})$ to $\rho(U(\mathcal{F}))$.

For the precise definition of a *regular set isomorphism T* and the induced mapping $\overline{T}$, see Lamperti (1958), Hardin (1981) or Wang and Stoev (2009). The following remark provides some intuition. Part (iii) is especially important since it shows that the two types of isometries can be identified.

Remark 2.1. (i) $\overline{U}$ is well defined in the sense that for any $r_i f_i \in \mathcal{R}_e(\mathcal{F}), i = 1, 2$ in (2.1), if $r_1 f_1 = r_2 f_2, \mu_1$-a.e., then $\overline{U}(r_1 f_1) = \overline{U}(r_2 f_2), \mu_2$-a.e. Similar result holds for $r_i f_i \in \mathcal{R}_{e,+}(\mathcal{F})$ as in (2.2).

(ii) T maps any two almost disjoint sets to almost disjoint sets. See Lamperti (1958).

(iii) The mapping $\overline{T}$ is both linear and max-linear, i.e., for $a, b \geq 0$,

$$\overline{T}(af + bg) = a\overline{T}f + b\overline{T}g \quad \text{and} \quad \overline{T}(af \vee bg) = a\overline{T}f \vee b\overline{T}g. \tag{2.4}$$

This follows from the definition $\overline{T}\mathbf{1}_A = \mathbf{1}_{T(A)}$ for measurable $A \subset S_1$ and the construction of $\overline{T}$ via simple functions. It is via $\overline{T}$ that the linearity and max-linearity are identified.

To make good use of (iii) in Remark 2.1, we introduce the notion of *positive-linearity*. We say a linear isometry U is *positive-linear*, if U maps all nonnegative functions to nonnegative functions. Accordingly, we say that $\mathcal{F} \subset L^\alpha_+(S, \mu)$ is a *positive-linear space*, if it is closed w.r.t. the metric $\rho_{\mu,\alpha}$ and all positive-linear combinations, i.e., for all $n \in \mathbb{N}, f_i \in \mathcal{F}, a_i \geq 0$, we have $g := \sum_{i=1}^n a_i f_i \in \mathcal{F}$. Note that the metric $(f, g) \mapsto \|f - g\|^{1 \wedge \alpha}_{L^\alpha(S,\mu)}$ restricted to $L^\alpha_+(S, \mu)$ generates the same topology as the metric $\rho_{\mu,\alpha}$. Clearly, Theorem 2.1 holds if $\mathcal{F}$ is a positive-linear (instead of a linear) subspace of $L^\alpha_+(S, \mu)$. In this case, $\overline{U}$ is also positive-linear. We conclude this section with the following refinement of statement (iii) in Remark 2.1.

Proposition 2.1. *Let U be as in Theorem 2.1. If $\mathcal{F}$ is a positive-linear subspace of $L^\alpha_+(S_1, \mu_1)$, then the linear isometry $\overline{U}$ in (2.3) is also a max-linear isometry from $\mathcal{R}_{e,+}(\mathcal{F})$ to $\mathcal{R}_{e,+}(U(\mathcal{F}))$. If $\mathcal{F}$ is a max-linear subspace of $L^\alpha(S_1, \mu_1)$, then the max-linear isometry $\overline{U}$ in (2.3) is also a positive-linear isometry from $\mathcal{R}_e(\mathcal{F})$ to $\mathcal{R}_e(U(\mathcal{F}))$.*

Proof. Suppose $\mathcal{F}$ is max-linear and $\overline{U}$ is a max-linear isometry. We show $\overline{U}$ is also positive-linear. First, if $\overline{U}$ in (2.3) is max-linear, then the mapping $\overline{T}$ from $L^\alpha_+(S_1, \rho(\mathcal{F}), \mu_1)$ to $L^\alpha_+(S_2, \rho(U(\mathcal{F})), \mu_2)$ is both max-linear and linear, by Remark 2.1(iii). Moreover, it is easily seen that $\overline{T}$ is positive-linear. Now, for $r_1 f_1, r_2 f_2 \in \mathcal{R}_{e,+}(\mathcal{F})$ as in (2.2), we have

$$\overline{U}(a_1 r_1 f_1 + a_2 r_2 f_2) = \overline{U}\left(\left(a_1 r_1 \frac{f_1}{f_1 \vee f_2} + a_2 r_2 \frac{f_2}{f_1 \vee f_2}\right)(f_1 \vee f_2)\right)$$

$$= \overline{T}\left(a_1 r_1 \frac{f_1}{f_1 \vee f_2} + a_2 r_2 \frac{f_2}{f_1 \vee f_2}\right)U(f_1 \vee f_2) = a_1 \overline{U}(r_1 f_1) + a_2 \overline{U}(r_2 f_2).$$

That is, $\overline{U}$ is positive-linear. The proof of the other case is similar, except that we need the existence of *full support function f* in $\mathcal{F}$, guaranteed by Lemma 3.2 in Hardin (1981). $\quad\square$

3. Identification of max-linear and positive-linear isometries

In this section we prove Theorem 1.1. It will be used to relate SαS and α-Fréchet processes in the next section. To do so, we need to introduce a subspace of $L^\alpha_+(S, \mu)$, which is closed w.r.t. the max-linear and positive-linear combinations. For any $F \subset L^\alpha_+(S, \mu)$, let

$$\mathcal{F}_+ := \overline{\text{span}_+}\{F\} \quad \text{and} \quad \mathcal{F}_\vee := \overline{\vee\text{-span}}\{F\} \tag{3.1}$$

denote the smallest positive-linear and max-linear subspace of $L^\alpha_+(S, \mu)$ containing the collection of functions F, respectively. We call them the max-linear and positive-linear spaces generated by F, respectively. (We also write $\mathcal{F} := \overline{\text{span}}\{F\}$ as the smallest linear subspace of $L^\alpha(S, \mu)$ containing F.) In general, we have $\mathcal{F}_+ \neq \mathcal{F}_\vee$. This means both $\mathcal{F}_+$ and $\mathcal{F}_\vee$ are too small to be closed w.r.t. both '$\sum$' and '$\bigvee$' operators. However, these two subspaces generate the same extended positive ratio space, on which the two types of isometries are identical. The following fact is proved in the Appendix.

Proposition 3.1. *Suppose $F \subset L^\alpha_+(S, \mu)$. Then $\mathcal{R}_{e,+}(\mathcal{F}_+) = \mathcal{R}_{e,+}(\mathcal{F}_\vee)$.*

Proof of Theorem 1.1. Let $F^{(i)} := \{f_1^{(i)}, \ldots, f_n^{(i)}\} \subset L^\alpha_+(S_i, \mu_i)$. We prove the 'if' part. Suppose Relation (1.6) holds and we will show (1.5). Relation (1.6) implies that there exists a unique max-linear isometry U from $\mathcal{F}_\vee^{(1)}$ onto $\mathcal{F}_\vee^{(2)}$, such that $U f_j^{(1)} = f_j^{(2)}, 1 \leq j \leq n$. Thus, Theorem 2.1 implies that the mapping

$$\overline{U} : \mathcal{R}_{e,+}(\mathcal{F}_\vee^{(1)}) \to \mathcal{R}_{e,+}(U(\mathcal{F}_\vee^{(1)}))$$

with form (2.3) is a max-linear isometry. By Proposition 2.1, we have that $\overline{U}$ is also a positive-linear isometry. By Proposition 3.1, $\overline{U}$ is a positive-linear isometry defined on $\mathcal{R}_{e,+}(\mathcal{F}_+^{(1)})$, which implies (1.5). The proof of the 'if' part is similar. $\quad\square$

To conclude this section, we will address the following question: for $f_1^{(1)}, \ldots, f_n^{(1)} \in L^\alpha(S_1, \mu_1)$, do there always exist nonnegative $f_1^{(2)}, \ldots, f_n^{(2)} \in L_+^\alpha(S_2, \mu_2)$ such that Relation (1.5) holds for any $a_j \in \mathbb{R}$? The answer is 'No'. As a consequence, in the next section we will see that there are SαS processes, which cannot be associated to any α-Fréchet process.

Proposition 3.2. *Consider $f_j^{(1)} \in L^\alpha(S_1, \mu_1)$, $1 \le j \le n$. Then, there exist some $f_j^{(2)} \in L_+^\alpha(S_2, \mu_2)$, $1 \le j \le n$ such that* (1.5) *holds, if and only if*

$$f_i^{(1)}(s)f_j^{(1)}(s) \ge 0, \mu_1\text{-a.e.} \quad \text{for all } 1 \le i, j \le n. \tag{3.2}$$

When (3.2) *is true, one can take $f_i^{(2)}(s) := |f_i^{(1)}(s)|$, $1 \le i \le n$ and $(S_2, \mu_2) \equiv (S_1, \mu_1)$ for* (1.5) *to hold.*

The proof is given in the Appendix. We will call (3.2) the *associable condition*.

4. Association of max- and sum-stable processes

In this section, by essentially applying Theorem 1.1, we associate an SαS process to every α-Fréchet process by Definition 1.1. The associated processes will be shown to have similar properties. However, we will also see that not all the SαS processes can be associated to α-Fréchet processes. We conclude with several examples.

Remark 4.1. In Definition 1.1, the associated SαS and α-Fréchet processes have the same $\alpha \in (0, 2)$. It is easy to see that, for any α-Fréchet process $\{Y_t\}_{t \in T}$ with spectral functions $\{f_t\}_{t \in T}$, $\{Y_t^\beta\}_{t \in T}$ is α/β-Fréchet with spectral functions $\{f_t^\beta\}_{t \in T}$, for all $0 < \alpha, \beta < \infty$. This transformation shows that the parameter α plays essentially no role in characterizing the dependence structure of the α-Fréchet process. Given an SαS process with nonnegative spectral functions, one could associate it to the 1-Fréchet process with spectral functions $\{f_t^\alpha\}_{t \in T}$. This leads to no loss of generality. Here, we chose to pair up the two α's for technical convenience.

The following result, a simple application of Theorem 1.1, shows the consistency of Definition 1.1, i.e., the notion of association is independent of the choice of the spectral functions.

Theorem 4.1. *Suppose an S α S process $\{X_t\}_{t \in T}$ and an α-Fréchet process $\{Y_t\}_{t \in T}$ are associated by $\{f_t^{(1)}\}_{t \in T} \subset L_+^\alpha(S_1, \mu_1)$. Then, $\{f_t^{(2)}\}_{t \in T} \subset L_+^\alpha(S_2, \mu_2)$ is a spectral representation of $\{X_t\}_{t \in T}$, if and only if it is a spectral representation of $\{Y_t\}_{t \in T}$. Namely,*

$$\left\{ \int_{S_1} f_t^{(1)} dM_{\alpha,+}^{(1)} \right\}_{t \in T} \stackrel{d}{=} \left\{ \int_{S_2} f_t^{(2)} dM_{\alpha,+}^{(2)} \right\}_{t \in T} \iff \left\{ \int_{S_1}^e f_t^{(1)} dM_{\alpha,\vee}^{(1)} \right\}_{t \in T} \stackrel{d}{=} \left\{ \int_{S_2}^e f_t^{(2)} dM_{\alpha,\vee}^{(2)} \right\}_{t \in T}, \tag{4.1}$$

where $M_{\alpha,+}^{(i)}$ and $M_{\alpha,\vee}^{(i)}$ are S α S random measures and α-Fréchet random sup-measures, respectively, on S_i with control measure μ_i, $i = 1, 2$.

As an immediate consequence, stationarity and self-similarity are preserved under association. Here we assume $T = \mathbb{R}^d$ or $\mathbb{Z}^d$.

Corollary 4.1. *Suppose an S α S process $\{X_t\}_{t \in T}$ and an α-Fréchet process $\{Y_t\}_{t \in T}$ are associated. Then,*

(i) *$\{X_t\}_{t \in T}$ is stationary if and only if $\{Y_t\}_{t \in T}$ is stationary.*
(ii) *$\{X_t\}_{t \in T}$ is self-similar with exponent H, if and only if $\{Y_t\}_{t \in T}$ is self-similar with exponent H.*

Proof. Suppose $\{X_t\}_{t \in T}$ and $\{Y_t\}_{t \in T}$ are associated by $\{f_t\}_{t \in T} \subset L_+^\alpha(S, \mu)$. (i) For any $h \in T$, letting $g_t = f_{t+h}, \forall t \in T$, by stationarity of $\{X_t\}_{t \in T}$, we obtain $\{g_t\}_{t \in T}$ as another spectral representation. Namely, $\{\int_S f_t dM_{\alpha,+}\}_{t \in T} \stackrel{d}{=} \{\int_S g_t dM_{\alpha,+}\}_{t \in T}$. By Theorem 4.1, the previous statement is equivalent to $\{{}^e\!\int_S f_t dM_{\alpha,\vee}\}_{t \in T} \stackrel{d}{=} \{{}^e\!\int_S g_t dM_{\alpha,\vee}\}_{t \in T}$, which is equivalent to the fact that $\{Y_t\}_{t \in T}$ is stationary. The proof of part (ii) is similar and thus omitted. $\square$

Observe that not all SαS processes can be associated to α-Fréchet processes, since not all SαS processes have nonnegative spectral representations. For an SαS process $\{X_t\}_{t \in T}$ with spectral representation $\{f_t\}_{t \in T}$ to have an associated α-Fréchet process, a necessary and sufficient condition is that for all $t_1, \ldots, t_n \in T, f_1, \ldots, f_{t_n}$ satisfy the associable condition (3.2). We say such SαS processes are *max-associable*. Now, Proposition 3.2 becomes:

Theorem 4.2. *Any S α S process $\{X_t\}_{t \in T}$ with representation (1.1) is max-associable, if and only if for all $t_1, t_2 \in T$,*

$$f_{t_1}(s)f_{t_2}(s) \ge 0, \quad \mu\text{-a.e.}. \tag{4.2}$$

Indeed, by Theorem 4.2 for any max-associable spectral representation $\{f_t\}_{t \in T}$, $\{|f_t|\}_{t \in T}$ is also a spectral representation for the same process. Clearly, if the spectral functions are nonnegative, then the SαS processes are max-associable. We give two simple examples next.

Example 4.1 (*Association of Mixed Fractional Motions*). Consider the self-similar SαS processes $\{X_t\}_{t \in \mathbb{R}_+}$ with the following representations

$$\{X_t\}_{t \in \mathbb{R}_+} \stackrel{d}{=} \left\{ \int\int_E \int_0^\infty t^{H - \frac{1}{\alpha}} g\left(x, \frac{u}{t}\right) M_{\alpha,+}(\mathrm{d}x, \mathrm{d}u)_{t \in \mathbb{R}_+} \right\}, \quad H \in (0, \infty), \tag{4.3}$$

where $(E, \mathcal{E}, \nu)$ is a standard Lebesgue space, $M_{\alpha,+}$ is an SαS random measure on $X \times \mathbb{R}_+$ with control measure $m(\mathrm{d}x, \mathrm{d}u) = \nu(\mathrm{d}x)\mathrm{d}u$ and $g \in L^\alpha(E \times \mathbb{R}_+, m)$. Such processes are called *mixed fractional motions* (see Burnecki et al., 1998). When $g \geq 0$ a.e., the process $\{X_t\}_{t \in \mathbb{R}_+}$ is max-associable. The Corollary 4.1 implies the associated α-Fréchet process is H-self-similar.

Example 4.2 (*Association of Chentzov SαS Random Fields*). Recall that $\{X_t\}_{t \in \mathbb{R}^n}$ is a Chentzov SαS random field, if

$$\{X_t\}_{t \in \mathbb{R}^n} \equiv \{M_{\alpha,+}(V_t)\}_{t \in \mathbb{R}^n} \stackrel{d}{=} \left\{ \int_S \mathbf{1}_{V_t}(u) M_{\alpha,+}(\mathrm{d}u) \right\}_{t \in \mathbb{R}^n}.$$

Here, $0 < \alpha < 2$, (S, μ) is a measure space and V_t, $t \in \mathbb{R}^n$ is a family of measurable sets such that $\mu(V_t) < \infty$ for all $t \in \mathbb{R}^n$ (see Ch. 8 in Samorodnitsky and Taqqu, 1994). Since $\mathbf{1}_{V_t}(u) \geq 0$, all Chentzov S$\alpha$S random fields are max-associable.

We conclude this section with some examples of SαS processes that are not max-associable. In particular, recall that the SαS processes with stationary increments (zero at $t = 0$) characterized by dissipative flows were shown in Surgailis et al. (1998) to have representation

$$\{X_t\}_{t \in \mathbb{R}} \stackrel{d}{=} \left\{ \int\int_E \int_{\mathbb{R}} (G(x, t + u) - G(x, u)) M_{\alpha,+}(\mathrm{d}x, \mathrm{d}u) \right\}_{t \in \mathbb{R}}. \tag{4.4}$$

Here, $(E, \mathcal{E}, \nu)$ is a standard Lebesgue space, $M_{\alpha,+}$, $\alpha \in (0, 2)$, is an SαS random measure with control measure $m(\mathrm{d}x, \mathrm{d}u) = \nu(\mathrm{d}x)\mathrm{d}u$ and $G : E \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that, for all $t \in \mathbb{R}$,

$$G_t(x, u) = G(x, t + u) - G(x, u), \quad x \in E, u \in \mathbb{R}$$

belongs to $L^\alpha(E \times \mathbb{R}, m)$. The process $\{X_t\}_{t \in \mathbb{R}}$ in (4.4) is called a *mixed moving average with stationary increments*. The following result provides a partial characterization of the max-associable SαS processes $\{X_t\}_{t \in T}$, which have the representation (4.4). We shall suppose that E is equipped with a metric ρ and endow $E \times \mathbb{R}$ with the product topology.

Proposition 4.1. *Consider an SαS process $\{X_t\}_{t \in \mathbb{R}}$ with representation (4.4). Suppose there exists a closed set $\mathcal{N} \subset E \times \mathbb{R}$, such that $m(\mathcal{N}) = 0$ and the function G is continuous at all $(x, u) \in \mathcal{N}^c := E \times \mathbb{R} \setminus \mathcal{N}$, w.r.t. the product topology. Then, $\{X_t\}_{t \in \mathbb{R}}$ is max-associable, if and only if*

$$G(x, u) = f(x)\mathbf{1}_{A_x}(u) + c(x), \quad on \ \mathcal{N}^c. \tag{4.5}$$

Namely, for all $x \in E$, $G(x, u)$ can take at most two values on $\mathcal{N}^c$.

Proof. By Theorem 4.2, $\{X_t\}_{t \in \mathbb{R}}$ is max-associable, if and only if for all $t_1, t_2 \in \mathbb{R}$,

$$G_{t_1}(x, u)G_{t_2}(x, u) = (G(x, t_1 + u) - G(x, u))(G(x, t_2 + u) - G(x, u)) \geq 0, \quad m\text{-a.e. } (x, u) \in E \times \mathbb{R}. \tag{4.6}$$

First, we show the 'if' part. Define $\widetilde{G}(x, u) := G(x, u)$ (given by (4.5)) on $\mathcal{N}^c$ and $\widetilde{G}(x, u) := f(x)\mathbf{1}_{A_x}(u) + c(x)$ on $\mathcal{N}$ (if A_x and $c(x)$ are not defined, then set $\widetilde{G}(x, u) = 0$). Set $\widetilde{G}_t(x, u) = \widetilde{G}(x, u + t) - \widetilde{G}(x, u)$. Note that $\widetilde{G}_t(x, u)$ is another spectral representation of $\{X_t\}_{t \in \mathbb{R}}$ and for all (x, u), $\left\{ \mathbf{1}_{A_x}(u + t) - \mathbf{1}_{A_x}(u) \right\}_{t \in \mathbb{R}}$ can take at most 2 values, one of which is 0. This observation implies (4.6) with $G_t(x, u)$ replaced by $\widetilde{G}_t(x, u)$, whence $\{X_t\}_{t \in \mathbb{R}}$ is max-associable.

Next, we prove the 'only if' part. We show that (4.6) is violated, if $G(x, u)$ takes more than 2 different values on $(\{x\} \times \mathbb{R}) \cap \mathcal{N}^c$ for some $x \in X$. Suppose there exist $x \in E$, $u_i \in \mathbb{R}$ such that $(x, u_i) \in \mathcal{N}^c$ and $g_{xi} := G(x, u_i)$ are mutually different, for $i = 1, 2, 3$. Indeed, without loss of generality we may suppose that $g_{x1} < g_{x2} < g_{x3}$. Then, by the continuity of G, there exists $\epsilon > 0$ such that $B_i := B(x, \epsilon) \times (u_i - \epsilon, u_i + \epsilon)$, $i = 1, 2, 3$ are disjoint sets with $B(x, \epsilon) := \{y \in E : \rho(x, y) < \epsilon\}$, ρ is the metric on E and

$$\sup_{B_1 \cap \mathcal{N}^c} G(x, u) < \inf_{B_2 \cap \mathcal{N}^c} G(x, u) \leq \sup_{B_2 \cap \mathcal{N}^c} G(x, u) < \inf_{B_3 \cap \mathcal{N}^c} G(x, u). \tag{4.7}$$

Put $t_1 = u_1 - u_2$ and $t_2 = u_3 - u_2$. Inequality (4.7) implies that $G_{t_1}(x, u)G_{t_2}(x, u) < 0$ on $B_2 \cap \mathcal{N}^c$. This, in view of Theorem 4.2, contradicts the max-associability. We have thus shown (4.5). $\square$

We give two classes of SαS processes, which cannot be associated to any α-Fréchet processes, according to Proposition 4.1.

Example 4.3 (*Non-associability of Linear Fractional Stable Motions*). The *linear fractional stable motions* (see Ch. 7.4 in Samorodnitsky and Taqqu, 1994) have the following spectral representations:

$$\{X_t\}_{t\in\mathbb{R}} \stackrel{\mathrm{d}}{=} \left\{ \int_{\mathbb{R}} \left[a \left((t+u)_+^{H-1/\alpha} - u_+^{H-1/\alpha} \right) + b \left((t+u)_-^{H-1/\alpha} - u_-^{H-1/\alpha} \right) \right] M_{\alpha,+}(\mathrm{d}u) \right\}_{t\in\mathbb{R}}.$$

Here $H \in (0, 1), \alpha \in (0, 2), H \neq 1/\alpha, a, b \in \mathbb{R}$ and $|a| + |b| > 0$. By Proposition 4.1, these processes are not max-associable.

Example 4.4 (*Non-associability of Telecom Processes*). The Telecom process offers an extension of fractional Brownian motion consistent with heavy-tailed fluctuations. It is a large scale limit of renewal reward processes and it can be obtained by choosing the distribution of the rewards accordingly (see Levy and Taqqu, 2000 and Pipiras et al., 2004). A Telecom process $\{X_t\}_{t\in\mathbb{R}}$ has the following representation

$$\{X_t\}_{t\in\mathbb{R}} \stackrel{\mathrm{d}}{=} \left\{ \int_{\mathbb{R}} \int_{\mathbb{R}} e^{(H-1)/\alpha} \left(F(e^s(t+u)) - F(e^s u) \right) M_{\alpha,+}(\mathrm{d}s, \mathrm{d}u) \right\}_{t\in\mathbb{R}},$$

where $1 < \alpha < 2, 1/\alpha < H < 1, F(z) = (z \wedge 0 + 1)_+, z \in \mathbb{R}$ and the SαS random measure $M_{\alpha,+}$ is with control measure $m_\alpha(\mathrm{d}s, \mathrm{d}u) = \mathrm{d}s\mathrm{d}u$. By Proposition 4.1, the Telecom process is not max-associable.

Remark 4.2. It is important that the index T in Proposition 4.1 is the entire real line $\mathbb{R}$. Indeed, in both Examples 4.3 and 4.4, when the time index is restricted to the half-line $T = \mathbb{R}_+$ (or $T = \mathbb{R}_-$), the processes $\{X_t\}_{t\in T}$ satisfy condition (4.2) and are therefore max-associable.

5. Association of classifications

In this section, we show how to apply the association technique to relate various classification results for SαS and α-Fréchet processes. Note that, many classifications of SαS (α-Fréchet as well) processes are induced by suitable decompositions of the measure space (S, μ). The following theorem provides an essential tool for translating classification results for SαS to α-Fréchet processes, and vice versa.

Theorem 5.1. *Suppose an SαS process $\{X_t\}_{t\in T}$ and an α-Fréchet process $\{Y_t\}_{t\in T}$ are associated by two spectral representations $\{f_t^{(i)}\}_{t\in T} \subset L_+^\alpha(S_i, \mu_i)$ for $i = 1, 2$. That is,*

$$\{X_t\}_{t\in T} \stackrel{\mathrm{d}}{=} \left\{ \int_{S_i} f_t^{(i)} \mathrm{d}M_{\alpha,+}^{(i)} \right\}_{t\in T} \quad and \quad \{Y_t\}_{t\in T} \stackrel{\mathrm{d}}{=} \left\{ {\int_{S_i}^e} f_t^{(i)} \mathrm{d}M_{\alpha,\vee}^{(i)} \right\}_{t\in T}, \quad i = 1, 2.$$

Then, for any measurable subsets $A_i \subset S_i, i = 1, 2$, we have

$$\left\{ \int_{A_1} f_t^{(1)} \mathrm{d}M_{\alpha,+}^{(1)} \right\}_{t\in T} \stackrel{\mathrm{d}}{=} \left\{ \int_{A_2} f_t^{(2)} \mathrm{d}M_{\alpha,+}^{(2)} \right\}_{t\in T} \Longleftrightarrow \left\{ {\int_{A_1}^e} f_t^{(1)} \mathrm{d}M_{\alpha,\vee}^{(1)} \right\}_{t\in T} \stackrel{\mathrm{d}}{=} \left\{ {\int_{A_2}^e} f_t^{(2)} \mathrm{d}M_{\alpha,\vee}^{(2)} \right\}_{t\in T}.$$

The proof follows from Theorem 1.1 by restricting the measures onto the sets $A_i, i = 1, 2$.

For an SαS process $\{X_t\}_{t\in T}$ with spectral functions $\{f_t\}_{t\in T} \subset L^\alpha(S, \mu)$, a decomposition typically takes the form $\{X_t\}_{t\in T} \stackrel{\mathrm{d}}{=} \{\sum_{j=1}^n X_t^{(j)}\}_{t\in T}$, where $X_t^{(j)} = \int_{A^{(j)}} f_t \mathrm{d}M_{\alpha,+}$ for all $t \in T$ and $A^{(j)}, 1 \leq j \leq n$ are disjoint subsets of $S = \bigcup_{j=1}^n A^{(j)}$. The components $\{X_t^{(j)}\}_{t\in T}, 1 \leq j \leq n$ are independent SαS processes. When $\{X_t\}_{t\in T}$ is max-associable, Theorem 5.1 enables us to define the *associated decomposition*, for the α-Fréchet process $\{Y_t\}_{t\in T}$ associated with $\{X_t\}_{t\in T}$. Namely, we have $\{Y_t\}_{t\in T} \stackrel{\mathrm{d}}{=} \{\bigvee_{j=1}^n Y_t^{(j)}\}_{t\in T}$, where $Y_t^{(j)} = {\int_{A^{(j)}}^e} |f_t| \mathrm{d}M_{\alpha,\vee}$ for all $t \in T$. Conversely, given a decomposition for α-Fréchet processes, we can define a corresponding decomposition for the associated SαS processes.

Example 5.1 (*Conservative–dissipative Decomposition*). In the seminal work, Rosiński (1995) established the *conservative–dissipative* decomposition for SαS processes. Namely, for any $\{X_t\}_{t\in T}$ with representation (1.1), one has

$$\{X_t\}_{t\in T} \stackrel{\mathrm{d}}{=} \{X_t^C + X_t^D\}_{t\in T},$$

where $X_t^C = \int_C f_t \mathrm{d}M_{\alpha,+}$ and $X_t^D = \int_D f_t \mathrm{d}M_{\alpha,+}$ for all $t \in T$, with C and D defined by

$$C := \left\{ s : \int_T f_t(s)^\alpha \lambda(\mathrm{d}t) = \infty \right\} \quad \text{and} \quad D := S \setminus C. \tag{5.1}$$

When $\{X_t\}_{t\in T}$ is stationary, the sets C and D correspond to the *Hopf decomposition* $S = C \cup D$ of the non-singular flow associated with $\{X_t\}_{t\in T}$ (see Rosiński, 1995 for details). Therefore, $\{X_t^C\}_{t\in T}$ and $\{X_t^D\}_{t\in T}$ are referred to as the *conservative* and *dissipative* components of $\{X_t\}_{t\in T}$, respectively. Theorem 5.1 enables us to use (5.1) to establish the parallel decomposition of the associated α-Fréchet process $\{Y_t\}_{t\in T}$. Namely, for the associated $\{Y_t\}_{t\in T}$, we have, $\{Y_t\}_{t\in T} \stackrel{\mathrm{d}}{=} \{Y_t^C \vee Y_t^D\}_{t\in T}$, where $Y_t^C = {\int_C^e} |f_t| \mathrm{d}M_{\alpha,\vee}$ and $Y_t^D = {\int_D^e} |f_t| \mathrm{d}M_{\alpha,\vee}$ for all $t \in T$. This decomposition was established in Wang and Stoev (2009) by using different tools.

Remark 5.1. Similar associations can be established for other decompositions, including *positive–null decomposition* (see Samorodnitsky, 2005 and Wang and Stoev, 2009), and the decompositions of the above two types for random fields ($T = \mathbb{Z}^d$ or $\mathbb{R}^d$; see Roy and Samorodnitsky, 2008 and Wang et al., 2009). A more specific decomposition for SαS processes with representation (4.4) was developed in Pipiras and Taqqu (2002), and one can obtain the corresponding decomposition for the associated α-Fréchet process by Theorem 5.1.

6. Discussion

Recently, Kabluchko (2009) introduced a similar notion of association. We became aware of his result toward the end of our work. The two approaches are technically different. Kabluchko's approach utilizes *spectral measures*, while ours is based on the structure of max-linear and linear isometries. These two approaches lead to the equivalent notions of association (see Lemma 2 in Kabluchko, 2009). For example, our Corollary 4.1 can also be obtained following his approach. On the other hand, our approach leads to a more direct proof of the following, which is Lemma 3 in Kabluchko (2009).

Lemma 6.1. *Let $\{X_t\}_{t \in T}$ be an S α S process and $\{Y_t\}_{t \in T}$ be an α-Fréchet process. Suppose $\{X_t\}_{t \in T}$ and $\{Y_t\}_{t \in T}$ are associated by $\{f_t\}_{t \in T} \subset L_+^\alpha(S, \mu)$. Then for any $t_1, t_2, \ldots \in T$, as $n \to \infty$, X_{t_n} converges in probability to X_t, if and only if Y_{t_n} converges in probability to Y_t.*

Proof. By Proposition 3.5.1 in Samorodnitsky and Taqqu (1994), $X_{t_n} \overset{p}{\to} X_t$ as $n \to \infty$, if and only if $\left\| f_{t_n} - f_t \right\|_{L_+^\alpha(S,\mu)}^\alpha \to 0$ as $n \to \infty$. This is equivalent to, by Theorem 2.1 and Lemma 2.3 in Stoev and Taqqu (2006), $Y_{t_n} \overset{p}{\to} Y_t$ as $n \to \infty$. □

Kabluchko (2009) also proved (Theorem 9 therein) that an α-Fréchet process is mixing (ergodic resp.) if and only if the associated SαS process is mixing (ergodic resp.). The proofs of these results, however, are not simple consequences of the notion of association. By association one can easily obtain new classes of α-Fréchet processes, while the probabilistic properties of the new processes (e.g. the associated α-Fréchet processes in examples in Section 4), however, do not automatically follow 'by association' and are yet to be investigated.

Acknowledgements

Appendix. Proofs of auxiliary results

We first need the following lemma.

Lemma A.1. *If $F \subset L_+^\alpha(S, \mu)$, then*

(i) $\rho(F) = \rho(\overline{\text{span}_+}(F)) = \rho(\overline{\vee\text{-span}}(F))$, *and*
(ii) *for any $f^{(1)} \in \overline{\text{span}_+}(F)$ and $f^{(2)} \in \overline{\vee\text{-span}}(F)$, $f^{(1)}/f^{(2)} \in \rho(F)$.*

Proof. (i) First, for any $f_i, g_i \in F$, $a_i \geq 0$, $b_i \geq 0$, $i \in \mathbb{N}$, we have

$$\left\{ \frac{\bigvee_{i \in \mathbb{N}} a_i f_i}{\bigvee_{j \in \mathbb{N}} b_j g_j} \leq x \right\} = \bigcap_{i \in \mathbb{N}} \left\{ \frac{a_i f_i}{\bigvee_{j \in \mathbb{N}} b_j g_j} \leq x \right\} = \bigcap_{i \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} \left\{ \frac{a_i f_i}{b_j g_j} < x + \frac{1}{k} \right\},$$

hence $\rho(\overline{\vee\text{-span}}(F)) \subset \rho(\overline{\text{span}_+}(F))$.

To show $\rho(\overline{\text{span}_+}(F)) \subset \rho(\overline{\vee\text{-span}}(F))$, we shall first prove that $\rho(\text{span}_+(F)) \subset \rho(\overline{\vee\text{-span}}(F))$, where $\text{span}_+(F)$ involves only finite positive-linear combinations. For all $f_1, f_2, g_1 \in F$, $a_1, b_1, b_2 \geq 0$, we have

$$\left\{ \frac{a_1 f_1 + a_2 f_2}{b_1 g_1} \leq x \right\} = \bigcup_{q_j \in \mathbb{Q}} \left(\left\{ \frac{a_1 f_1}{b_1 g_1} \leq q_j \right\} \cap \left\{ \frac{a_2 f_2}{b_1 g_1} \leq x - q_j \right\} \right).$$

This shows that $(a_1 f_1 + a_2 f_2)/b_1 g_1$ is $\rho(\overline{\vee\text{-span}}(F))$ measurable. By using the fact that F contains only nonnegative functions and since $\left\{ \frac{b_1 g_1}{a_1 f_1 + a_2 f_2} \leq x \right\} = \left\{ \frac{a_1 f_1 + a_2 f_2}{b_1 g_1} \geq \frac{1}{x} \right\}$, for $x > 0$, we similarly obtain that $(a_1 f_1 + a_2 f_2)/(b_1 g_1 + b_2 g_2)$ is $\rho(\overline{\vee\text{-span}}(F))$ measurable. Similarly arguments can be used to show that $(\sum_{i=1}^n a_i f_i)/(\sum_{i=1}^n b_i g_i)$ is $\rho(\overline{\vee\text{-span}}(F))$ measurable for all $a_i, b_i \geq 0$, $f_i, g_i \in F$, $1 \leq i \leq n$.

We have thus shown that $\rho(\text{span}_+(F)) \subset \rho(\overline{\vee\text{-span}}(F))$. If now $f, g \in \overline{\text{span}_+}(F)$, then there exist two sequences $f_n, g_n \in \text{span}_+(F)$, such that $f_n \to f$ and $g_n \to g$ a.e.. Thus, $h_n := f_n/g_n \to h := f/g$ as $n \to \infty$, a.e.. Since h_n are $\rho(\text{span}_+(F))$ measurable for all $n \in \mathbb{N}$, so is h. Hence $\rho(\overline{\text{span}_+}(F)) = \rho(\text{span}_+(F)) \subset \rho(\overline{\vee\text{-span}}(F))$.

(ii) By the previous argument, it is enough to focus on finite linear and max-linear combinations. Suppose $f^{(1)} = \sum_{i=1}^{n} a_i f_i$ and $f^{(2)} = \bigvee_{j=1}^{p} b_j g_j$ for some $f_i, g_j \in F$, $a_i, b_j \geq 0$, $1 \leq i \leq n$, $1 \leq j \leq p$. Then, for all $x > 0$,

$$\left\{ \frac{\sum_{i=1}^{n} a_i f_i}{\bigvee_{j=1}^{p} b_j g_j} < x \right\} = \bigcup_{j=1}^{p} \left\{ \sum_{i=1}^{n} a_i \frac{f_i}{g_j} < x b_j \right\} \in \rho(F).$$

It follows that $f^{(1)} / f^{(2)} \in \rho(F)$. □

Proof of Proposition 3.1. First we show $\mathcal{R}_{e,+}(\mathcal{F}_\vee) \supset \mathcal{R}_{e,+}(\mathcal{F}_+)$, where $\mathcal{F}_\vee$ and $\mathcal{F}_+$ are defined in (3.1). By (2.2), it suffices to show that, for any $r_2 \in \rho(\mathcal{F}_+), f^{(2)} \in \mathcal{F}_+$, there exist $r_1 \in \rho(\mathcal{F}_\vee)$ and $f^{(1)} \in \mathcal{F}_\vee$, such that

$$r_1 f^{(1)} = r_2 f^{(2)}. \tag{A.1}$$

To obtain (A.1), we need the concept of *full support*. We say a function g has full support in F (an arbitrary collection of functions defined on (S, μ)), if $g \in F$ and for all $f \in F$, $\mu(\text{supp}(g) \setminus \text{supp}(f)) = 0$. Here $\text{supp}(f) := \{s \in S : f(s) \neq 0\}$. By Lemma 3.2 in Wang and Stoev (2009), there exists function $f^{(1)} \in \mathcal{F}_\vee$, which has full support in $\mathcal{F}_\vee$. One can show that this function has also full support in $\mathcal{F}_+$. Indeed, let $g \in \mathcal{F}_+$ be arbitrary. Then, there exist $g_n = \sum_{i=1}^{k_n} a_{ni} g_{ni}, a_{ni} \geq 0$ and $g_{ni} \in F \subset \mathcal{F}_\vee$ such that $g_n \xrightarrow{\mu} g$ as $n \to \infty$. Note that $\mu(\text{supp}(g_n) \setminus \text{supp}(f)) = 0$ for all n. Thus, for all $\epsilon > 0$, we have $\mu(|g_n - g| > \epsilon) \geq \mu(\{|g| > \epsilon\} \setminus \text{supp}(f))$. Since $\mu(\{|g_n - g| > \epsilon) \to 0$ as $n \to \infty$, it follows that $\mu(\{|g| > \epsilon\} \setminus \text{supp}(f)) = 0$ for all $\epsilon > 0$, i.e., $\mu(\text{supp}(g) \setminus \text{supp}(f)) = 0$. We have thus shown that f has full support in $\mathcal{F}_+$.

Now, set $r_1 := r_2 \left(f^{(2)} / f^{(1)} \right)$, we have (A.1). (Note that $f^{(2)} = 0$, μ-a.e. on $S \setminus \text{supp}(f^{(1)})$. By setting $0/0 = 0, f^{(2)} / f^{(1)}$ is well defined.) Lemma A.1 (ii) implies that $f^{(2)} / f^{(1)} \in \rho(F)$, whence $r_1 \in \rho(F) = \rho(\mathcal{F}_+)$. We have thus shown $\mathcal{R}_{e,+}(\mathcal{F}_\vee) \supset \mathcal{R}_{e,+}(\mathcal{F}_+)$. In a similar way one can show $\mathcal{R}_{e,+}(\mathcal{F}_\vee) \subset \mathcal{R}_{e,\vee}(\mathcal{F}_+)$. □

Proof of Proposition 3.2. First, suppose (3.2) does not hold but (1.5) holds. Then, without loss of generality, we can assume that there exists $S_0^{(1)} \subset S_1$ such that $f_1^{(1)}(s) > 0, f_2^{(1)}(s) < 0$ for all $s \in S_0^{(1)}$ and $\mu(S_0^{(1)}) > 0$. It follows from (1.5) that there exists a linear isometry U such that, by Theorem 2.1, $U f_i^{(1)} = f_i^{(2)} = \overline{T}(r_i) U(f)$, with certain f and $r_i = f_i^{(1)} / f$, for $i = 1, 2$. In particular, f can be taken with full support. Note that $\text{sign}(r_1) \neq \text{sign}(r_2)$ on $S_0^{(1)}$. It follows that $f_1^{(2)}$ and $f_2^{(2)}$ have different signs on a set of positive measures (indeed, this set is the image of the $S_0^{(1)}$ under the regular set isomorphism T). This contradicts the fact that $f_1^{(2)}$ and $f_2^{(2)}$ are both nonnegative on S_2.

On the other hand, suppose (3.2) is true. Define $U f_i^{(1)} := |f_i^{(1)}|$. It follows from (3.2) that U can be extended to a positive-linear isometry from $L^\alpha(S_1, \mu_1)$ to $L_+^\alpha(S_2, \mu_2)$, which implies (1.5). □

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