Statistics 406 Lab 6
GSI: Adam Rothman
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1. Homework 5 problem 4 clarification:

   Let $X_1, \ldots, X_n$ be iid $U(0, K)$. We estimated K using $\hat{K} = \max(X_1, \ldots, X_n)$. I will derive the $E\hat{K}$, $\mathrm{Var}\hat{\mathrm{K}}$, and explain why $n\mathrm{Var}\hat{\mathrm{K}} \to 0$.

   To start, we find the density for $\hat{K}$:

$$\begin{aligned} P(\hat{K} \leq t) &= P(X_1 < t) \cdots P(X_n < t) \\ &= (t/K)^n \\ f_{\hat{K}}(t) &= nt^{n-1}/K^n \end{aligned}$$

   Now we find $E\hat{K}$ and $E\hat{K}^2$

$$\begin{aligned} E\hat{K} &= \int_0^K \frac{nt^n}{K^n} dt \\ &= nK/(n+1) \\ E\hat{K}^2 &= \int_0^K \frac{nt^{n+1}}{K^n} dt \\ &= nK^2/(n+2) \\ \mathrm{Var}\hat{K} &= E\hat{K}^2 - (E\hat{K})^2 \\ &= K^2 \frac{n}{n^3 + 4n^2 + 5n + 2} \end{aligned}$$

   Thus

$$n\mathrm{Var}\hat{K} = K^2 \frac{n^2}{n^3 + 4n^2 + 5n + 2} \to 0$$

2. We observed paired measurements (X_i, Y_i) on n subjects. $X_1, \ldots, X_n$ are iid from a population with mean μ_X and $Y_1, \ldots, Y_n$ are iid from a population with mean μ_Y. An example of paired measurements are before and after measurements of an individual's weight. For the purposes of this problem let:

$$\begin{aligned} X_i &= \mu_X + T_i + \epsilon_{X_i} \\ Y_i &= \mu_Y + T_i + \epsilon_{Y_i} \end{aligned}$$

Where T_i are iid $N(0, \tau^2)$ and ϵ_{X_i} and ϵ_{Y_i} are iid $N(0, 1)$. Use simulation to estimate the powers of the usual two-sample t-test (althought this is obviosly a paired t-test situation), and the one sample t-test on the differences for the alternative hypothesis $\mu_X > \mu_Y$. Use $n = 10$, $\mu_X = 1$, $\mu_Y = 0$, and τ^2 ranging from 0 to 2 by 0.2.

```
Output <- NULL
for (v in seq(0,2,0.2))
{
  #Initialize the array that will store 2x1000 powers
  Q <- array(0, c(1000,2))

  #Repeat the process 1000 times
  for (k in (1:1000))
  {
    #Generate the X and Y data sequences
    T <- sqrt(v)*rnorm(10)
    X <- rnorm(10) + T + 1
    Y <- rnorm(10) + T

    #Compute what goes into formulas
    VX <- var(X)
    VY <- var(Y)
    DM <- mean(X) - mean(Y)
    Sp2 <- (9*VX + 9*VY) / 18

    ## Two-sample (unpaired) statistic.
    TS2 <- DM/sqrt(20*Sp2/100)
    D <- X-Y
    V <- var(D)

    ## One-Sample (paired) statistic
    TS1 <- sqrt(10)*mean(D) / sqrt(V)

    ## Compute the powers for each test and store in Q
    Q[k,1] <- (TS1 > qt(0.95, 9))
    Q[k,2] <- (TS2 > qt(0.95, 18))
  }
Output <- rbind( Output, c(v, colMeans(Q)) )
}
print(Output)
```