Estimating Stochastic Bounds for Multivariate Tail Dependence

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Max-linear models

- Let $Z_1, Z_2, \dots$ be heavy-tailed "shocks" to a system of components $D = \{1, \dots, d\}$.
- Consider weights $\theta_{jk} > 0$, $j = 1, \dots, p$, such that

$$X_j = \bigvee_{k=1}^{p} \theta_{jk} Z_k, \tag{1}$$

measures the peak stress on component j, due to shocks $Z_1, Z_2, \dots$

  - $Z_1, Z_2, \dots$ are iid with $\mathbb{P}(Z_1 \le z) = e^{-1/z}$.
  - Weights sum to unity: $\sum_{k=1}^{p} \theta_{jk} = 1, \; j = 1, \dots, d$.
  - $X_j \overset{d}{=} Z_1$ for all $j = 1, \dots, d$.

- Models of type (1) are frequently encountered in insurance, finance, and reliability, as models for dependence under **worst case scenaria**.
- The max-linear equation (1) can be expressed in matrix notation

$$\mathbf{X} = \Theta \circledcirc \mathbf{Z}, \tag{2}$$

where $\mathbf{X} = (X_1, \dots, X_d)^\top, \mathbf{Z} = (Z_1, \dots, Z_p)^\top$, and Θ is the $d \times p$ matrix with entries θ_{jk}. The *max-linear* operator $\circledcirc$ performs matrix multiplication with sum replaced by max.

Characterizing tail dependence

- Distribution function:

$$F_\theta(\mathbf{x}) := \mathbb{P}(\mathbf{X} \le \mathbf{x}) = \exp\left[-\sum_{k=1}^{p} \bigvee_{j=1}^{d} \theta_{jk}/x_j \right].$$

- Tail exponent function:

$$V_\theta(\mathbf{x}) := -\log F_\theta(\mathbf{x}) = \sum_{k=1}^{p} \bigvee_{j=1}^{d} \theta_{jk}/x_j.$$

Extremal coefficient function

Let $J \subset D$. A popular summary measure for tail dependence is the *extremal coefficient function*

$$\vartheta(J) := \sum_{k=1}^{p} \bigvee_{j \in J} \theta_{jk}.$$

$\vartheta : 2^D \to [1, |D|]$, is roughly the effective number of independent variables in $\{X_j, j \in J\}$, for all $J \in 2^D$

$$\mathbb{P}(X_j \le x, j \in J) = \mathbb{P}(X_1 \le x)^{\vartheta(J)}.$$

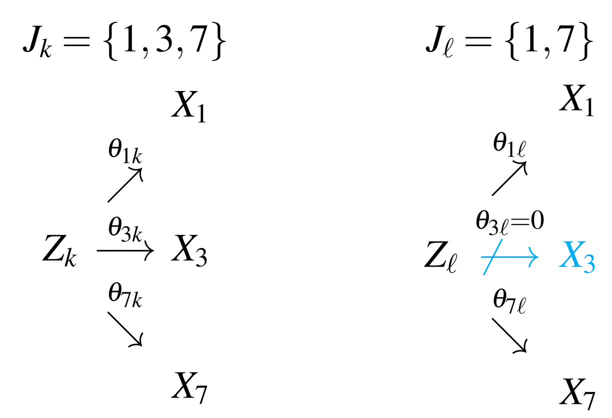
Inference problem

- Estimation of θ is a difficult problem.
- p maybe unknown or infinite.
- No likelihood for $d > 2$, no MLE or Bayesian inference.
- **With respect to worst case scenario, estimating upper bounds is a viable alternative.**

Upper bound model (UBM)

- Assume the following:

(A1) **Power set factors**: Exactly one Z_k effects a single subset J_k in the power set of the system $\{1, \dots, d\}$.
Example:



(A2) **Homogeneity**: $\theta_{1k} = \theta_{3k} = \theta_{7k} = \beta_k$, $\theta_{1\ell} = \theta_{3\ell} = \beta_\ell$.

- Let Ψ be the $d \times p$ binary matrix whose columns correspond to the support of $J_k, k = 1, \dots, p = 2^d - 1$.

$$\Psi := \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & \cdots & 0 & 1 \end{bmatrix}_{d \times p}$$

- Under (A1) and (A2), the model (2) becomes

$$\tilde{\mathbf{X}} = \Psi \circledcirc (\mathbf{Z} \circ \beta), \tag{3}$$

where $\beta = (\beta_1, \dots, \beta_p)^\top$ and $\circ$ is element-wise multiplication.

Properties of the UBM

- Tail exponent function

$$\tilde{V}_\beta(\mathbf{x}) = \left(\mathbf{x}^{-\top} \circledcirc \Psi\right)\beta,$$

where $\mathbf{x}^{-\top} = \left(\frac{1}{x_1}, \dots, \frac{1}{x_d}\right)$.

- Extremal coefficient function

$$\tilde{\vartheta}(D) = \|\beta\|_1.$$

- (Strokorb and Schlather 2013): If $\mathbf{X}$ is a max-linear model of type (2) with $\vartheta(J) = \tilde{\vartheta}(J)$ for all $J \subset D$, then

$$\mathbb{P}(\mathbf{X} > \mathbf{x}) \le \mathbb{P}(\tilde{\mathbf{X}} > \mathbf{x}).$$

- Induced graph structure

$$G = \Psi \mathrm{diag}(\beta) \Psi^\top,$$

$G_{ij} = 0$ implies $\tilde{X}_i$ and $\tilde{X}_j$ are independent.

- Model constraints on β

(C1) Non-negative: $\beta \in \mathbb{R}_+^p$.

(C2) L_1bound: $1 \le \|\beta\|_1 \le d$.

(C3) Standard margins: $\Psi\beta = \mathbf{1}$.

Estimation for UBM

- Observing iid $\mathbf{X}_1, \dots, \mathbf{X}_n$, estimate β.
- Number of parameters $p = 2^d - 1 \gg n$.
- No tractable likelihood.

An M-estimator for max-linear models

- Let μ be a measure on $\left(\mathbb{R}_+^d, \mathcal{B}\left(\mathbb{R}_+^d\right)\right)$ and define

$$\hat{\beta}_n = \arg\min_{\beta \in B} \int_{\mathbb{R}_+^d} \{\exp[-V_\beta(\mathbf{x})] - F_n(\mathbf{x})\}^2 \mu(d\mathbf{x})$$

  - $F_n(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{\{\mathbf{X}_i \le \mathbf{x}\}}$.
  - B is a feasible region defined by constraints C1- C3.

- If μ is discrete with atoms $\mathbf{x}_1, \dots, \mathbf{x}_M$ having equal mass, then

$$\hat{\beta}_n = \arg\min_{\beta \in B} \|\mathbf{f}_\beta - \mathbf{f}_n\|^2, \tag{4}$$

where

  - $\mathbf{f}_\beta = \left(\exp[-V_\beta(\mathbf{x}_i)], i = 1, \dots, M\right)^\top$.
  - $\mathbf{f}_n = \left(F_n(\mathbf{x}_i), i = 1, \dots, M\right)^\top$.

- (Yuen and Stoev 2013): Under mild regularity, $\hat{\beta}_n$ in (4) is a consistent estimator.

Simulation

- We simulate $n = 50$ iid realizations from the UBM with $d = 7 \implies p = 2^d - 1 = 127$.

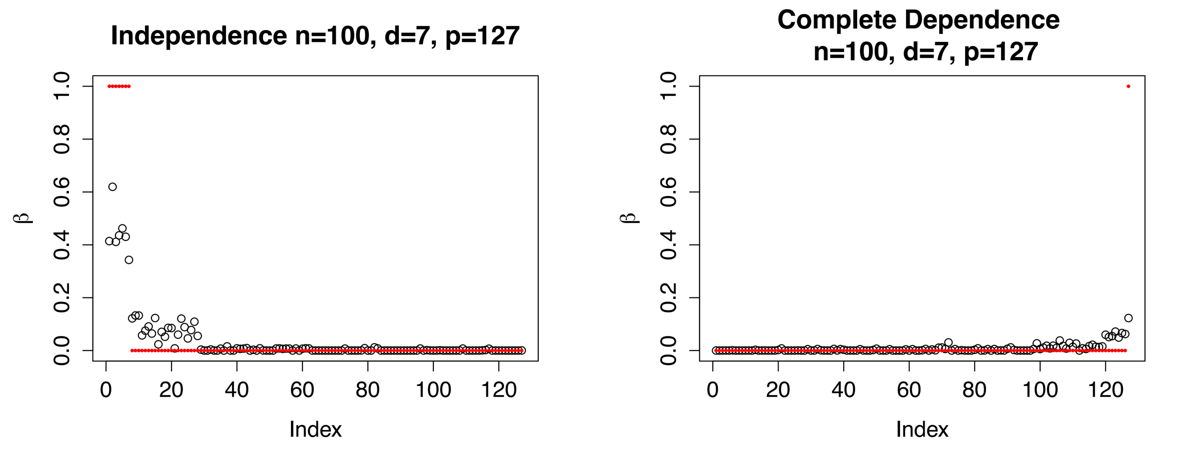


Figure 1: *Circles indicate estimates of $\hat{\beta}_n$ under independence (left) and complete dependence (right). Red dots indicate true β.*

For details on this research and other projects please visit:

http://www.stat.lsa.umich.edu/~bobyuen

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