Homework 7

1. Suppose $\mathbf{x}$ and $\mathbf{y}$ are jointly multivariate normal, with means μ_x and μ_y, variances Σ_x and Σ_y and $\text{Cov}(\mathbf{x}, \mathbf{y}) = \Sigma_{xy}$. Suppose Σ_x and Σ_y are invertible.

   (i) Let $\mathbf{z} = \mathbf{x} - \Sigma_{xy}\Sigma_y^{-1}\mathbf{y}$. Show that $\mathbf{z}$ and $\mathbf{y}$ are uncorrelated. Explain why $\mathbf{z}$ and $\mathbf{y}$ are therefore independent.

   (ii) Use Part (i), together with the properties of conditional expectation from the lecture notes (Lecture 9), to show that

   $$E[\mathbf{x}|\mathbf{y}] = \mu_x + \Sigma_{xy}\Sigma_y^{-1}(\mathbf{y} - \mu_y)$$

   (iii) Show that $\text{Var}(\mathbf{x}|\mathbf{y}) = \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{yx}$ (usually conditional variances depend on the value of $\mathbf{y}$, but for conditional multivariate normal random variables this is not the case).

2. Suppose x and y are independent univariate complex normal $x \sim N^c(0, \sigma^2)$, $y \sim N^c(0, \sigma^2)$. Let $\mathbf{z} = (z_1, z_2)'$ be defined by

   $$\begin{aligned} z_1 &= a_{11}x + a_{12}y \\ z_2 &= a_{21}x + a_{22}y \end{aligned}$$

   where a_{ij} are complex valued numbers. Use the definition of the multivariate complex normal distibution to find the distribution of $\mathbf{z}$. Specifically, show that $\mathbf{z}$ is a bivariate complex normal,

   $$\mathbf{z} \sim N^c(\mathbf{0}, \sigma^2 A\bar{A}')$$

   where $A = [a_{ij}]$. Hint: Write $A = B + iC$ where B and C are real matrices. (note: $\bar{A}'$, the transpose complex conjugate, is sometimes written A^*).