Homework 9, due 4/8

A state space model has state process x_t and observation process y_t. Let x_t take values in a state space $\mathcal{X}$, and y_t take values in the observation space $\mathcal{Y}$. Recall that, by definition, for a state space model:

$$\begin{aligned} p(x_t|x_{t-1}, x_{t-2}, \ldots, y_{t-1}, y_{t-2}, \ldots) &= p(x_t|x_{t-1}), \\ p(y_t|x_t, x_{t-1}, \ldots, y_{t-1}, y_{t-2}, \ldots) &= p(y_t|x_t). \end{aligned}$$

Here, for example, $p(y_t|x_t)$ means the conditional density of y_t given x_t. This makes use of a convenient abuse of notation, where we use the time series both as labels and as the variables. Prove the following recursions:

$$p(x_t|y_1, \ldots, y_{t-1}) = \int_{\mathcal{X}} p(x_{t-1}|y_1, \ldots, y_{t-1}) p(x_t|x_{t-1}) dx_{t-1}, \tag{1}$$

$$p(x_t|y_1, \ldots, y_t) = \frac{p(x_t|y_1, \ldots, y_{t-1}) p(y_t|x_t)}{p(y_t|y_1, \ldots, y_{t-1})}, \tag{2}$$

$$p(x_t|y_1, \ldots, y_T) \propto p(y_t, \ldots, y_T|x_t) p(x_t|y_1, \ldots, y_{t-1}), 1 < t < T \tag{3}$$

Note: (1) and (2) are the forward filtering/prediction recursions. (3) shows how to use forward and backward recursions to solve the smoothing problem.

Note: this is the last homework; afterwards will remain only the final project. There will not be a homework on implementing state space models, but looking at the basic structural model should still be accessible for the final project, if appropriate.