

STATS 531 / ECON 677 WINTER 09
HOMEWORK 2

PROBLEM 2.8 (A) That the variability in the glacial varve record changes over time can be checked over many different intervals. From the plot of x_t , this change is very pronounced when comparing the intervals [100,200] and [375,475] where the variance appears to be roughly constant at very low and high values respectively. In fact, $\hat{\gamma}_{x_{100:200}}(0) = 35$ and $\hat{\gamma}_{x_{375:475}}(0) = 497$ whereas $\hat{\gamma}_{y_{100:200}}(0) = 0.17 \approx \hat{\gamma}_{y_{375:475}}(0) = 0.23$. The transformation stabilizes the variance for the whole series, not only these intervals, as can be seen from the plot of y_t . The histogram of x_t is skewed to the right and the histogram of y_t is rather symmetric and a lot closer to the desired bell shape.

(B) Looking at the plot of $\hat{\rho}_y(h)$ we see that the values slowly decrease until about $h = 125$, where they become basically 0, and peak at negative values at somewhere between $h = 200$ and $h = 250$ before reverting towards 0 again. This sample ACF reflects the periodicities that we see in the plot of y_t , where one could hypothesize that there is a cycle that takes about 500 years or so to complete (corresponding to the 250 and 125 observed in $\hat{\rho}_y(h)$). Comparing with Fig 1.2, we see some evidence of increasing trend in parts of the series.

(C) u_t appears to be stationary since both the mean and the variance seem to be constant over time (unlike y_t , which showed the cycle described in **(B)**). The sample ACF is close to zero except for $h = 1$ which, because it resembles the white noise ACF, suggests stationarity again. The first difference of the log transformed annual series is the annual growth rate, assuming exponential growth, i.e.

$$\begin{aligned}u_t &= y_t - y_{t-1} = \ln\left(\frac{x_t}{x_{t-1}}\right) \\e^{u_t} &= \frac{x_t}{x_{t-1}} \\x_t &= e^{u_t} x_{t-1}.\end{aligned}$$

This can also be interpreted as an approximation to the annual percentage change of the

series, $p_t = \frac{x_t - x_{t-1}}{x_{t-1}}$, using a Taylor series approximation, i.e. for $|p_t| < 1$,

$$\begin{aligned}\ln\left(\frac{x_t}{x_{t-1}}\right) &= \ln\left(\frac{x_t - x_{t-1} + x_{t-1}}{x_{t-1}}\right) \\ &= \ln(p_t + 1) \\ &= p_t + o(p_t).\end{aligned}$$

(D) The theoretical ACF of the model in Example 1.23 is zero for lags larger than $h = 1$. Since the sample ACF of u_t is very similar to this it makes sense to try a similar model where the white noise can be something else than only 1 or -1. To derive the covariance function of the process, note that

$$\begin{aligned}E[u_{t+h}u_t] &= E[(\mu_{ut} + w_{t+h} - \theta w_{t+h-1})(\mu_{ut} + w_t - \theta w_{t-1})] \\ &= E[\mu_{u(t+h)}\mu_{ut} + \mu_{u(t+h)}w_t - \mu_{u(t+h)}\theta w_{t-1} + \\ &\quad + w_{t+h}\mu_{ut} + w_{t+h}w_t - w_{t+h}\theta w_{t-1} - \\ &\quad - \theta w_{t+h-1}\mu_{ut} - \theta w_{t+h-1}w_t + \theta^2 w_{t+h-1}w_{t-1}]\end{aligned}$$

and that, since we assume stationarity $\mu_{ut} = \mu_{u(t+h)} = \mu_u$ and $\gamma_u(h) = E[u_{t+h}u_t] - \mu_u^2$,

$$\begin{aligned}E[u_{t+h}u_t] &= \mu_u^2 + E[w_{t+h}w_t] - \theta E[w_{t+h}w_{t-1}] - \theta E[w_{t+h-1}w_t] - \theta^2 E[w_{t+h-1}w_{t-1}] \\ \gamma_u(0) &= \sigma_w^2 + \theta^2 \sigma_w^2 \\ \gamma_u(1) &= -\theta \sigma_w^2 \\ \gamma_u(2) &= 0\end{aligned}$$

and $\gamma_u(h) = 0$ for $h > 2$. This, together with $\gamma_u(h) = \gamma_u(-h)$, shows the result. Equating the sample ACF to the theoretical values gives the moment equations

$$\begin{aligned}\hat{\gamma}_u(0) &= \hat{\sigma}_w^2(1 + \hat{\theta}) \\ \hat{\gamma}_u(1) &= -\hat{\theta}\hat{\sigma}_w^2,\end{aligned}$$

and the estimators from the method of moments

$$\begin{aligned}\hat{\sigma}_w^2 &= -\frac{\hat{\gamma}_u(1)}{\hat{\theta}} \\ \hat{\theta} &= \frac{-\hat{\gamma}_u(0) \pm \sqrt{\hat{\gamma}_u(0)^2 - 4\hat{\gamma}_u(1)^2}}{2\hat{\gamma}_u(1)}.\end{aligned}$$

