STATS 531 / ECON 677 WINTER 09
HOMEWORK 3

<u>**PROBLEM 3.1**</u> For any $\theta \in \mathbb{R}$,

$$\begin{aligned}(1-|\theta|)^2 &\geq 0\\ 1+|\theta|^2 &\geq 2|\theta|.\end{aligned}$$

Since

$$\begin{aligned}|\rho_x(1)| &= \left|\frac{\theta}{1+\theta^2}\right|\\ &= \frac{|\theta|}{1+|\theta|^2}\\ &\leq \frac{|\theta|}{2|\theta|},\end{aligned}$$

it follows that $|\rho_x(1)| \leq \frac{1}{2}$ for any $\theta \in \mathbb{R}$. Let $\theta^\star$ be the extrema of $\rho_x(1)$. Note that

$$\begin{aligned}\frac{d}{d\theta}\rho_x(1) &= \frac{1 \times (1+\theta^2) - \theta \times 2\theta}{(1+\theta^2)^2}\\ &= \frac{1-\theta^2}{(1+\theta^2)^2}.\end{aligned}$$

Setting $\frac{d}{d\theta}\rho_x(1)$ equal to zero, $1-\theta^{\star 2} = 0$ and it follows that $\theta^\star = \pm 1$ are extrema. Since $\frac{d}{d\theta}\rho_x(1) \neq 0$ for $\theta \notin \{-1, 1\}$ and it exists for all $\theta \in \mathbb{R}$, these are unique global extrema. The second order conditions would show whether these are maxima or minima. Since we know that $|\rho_x(1)| \leq \frac{1}{2}$, plugging $\theta^\star$ into $\rho_x(1)$ shows directly that $\theta = 1$ and $\theta = -1$ are a maximum and minimum respectively.

<u>**PROBLEM 3.2**</u>

<u>**(A)**</u> Note that

$$\begin{aligned}x_1 &= w_1\\ x_2 &= \phi x_1 + w_2 = \phi w_1 + w_2\\ x_3 &= \phi x_2 + w_3 = \phi(\phi w_1 + w_2) + w_3 = \phi^2 w_1 + \phi w_2 + w_3\\ &\vdots\end{aligned}$$

so that, for $t = 1, 2, \ldots$ and $h \geq 0$,

$$\begin{aligned}
x_t &= \sum_{i=0}^{t-1} \phi^i w_{t-i} \\
\mu_x &= \sum_{i=0}^{t-1} \phi^i E[w_{t-i}] = 0 \\
cov(x_t, x_{t-h}) &= E[\sum_{i=0}^{t-1} \phi^i w_{t-i} \sum_{j=0}^{t-h-1} \phi^j w_{t-h-j}] \\
&= \sum_{i=0}^{t-1} \sum_{j=0}^{t-h-1} \phi^i \phi^j E[w_{t-h-j} w_{t-i}] \\
&= \sum_{j=0}^{t-h-1} \phi^j \phi^{h+j} E[w_{t-h-j} w_{t-h-j}] \\
&= \phi^h \sigma_w^2 \sum_{j=0}^{t-h-1} \phi^{2j} = \phi^h \sigma_w^2 \frac{1 - \phi^{2(t-h)}}{1 - \phi^2},
\end{aligned}$$

since the expectation inside the summation is σ_w^2 for $i = h + j$ and $h \geq 0$ and zero otherwise; and since $\phi^2 \leq |\phi| < 1$ which implies $\sum_{j=0}^{\infty} (\phi^2)^j = \frac{1}{1-\phi^2}$, $\sum_{j=t-h}^{\infty} (\phi^2)^j = \sum_{k=0}^{\infty} (\phi^2)^{k+t-h} = \phi^{2(t-h)} \sum_{k=0}^{\infty} (\phi^2)^k = \frac{\phi^{2(t-h)}}{1-\phi^2}$ and

$$\sum_{j=0}^{t-h-1} (\phi^2)^j = \sum_{j=0}^{\infty} (\phi^2)^j - \sum_{j=t-h}^{\infty} (\phi^2)^j = \frac{1 - \phi^{2(t-h)}}{1 - \phi^2}.$$

The variance of x_t is $cov(x_t, x_t) = \sigma_w^2 \frac{1-\phi^{2t}}{1-\phi^2}$. Since it depends on t for $\phi \neq 0$, the process is in general non-stationary.

(B) From **(A)**, we have that for $h \geq 0$

$$\begin{aligned}
var(x_{t-h}) &= \sigma_w^2 \frac{1 - \phi^{2(t-h)}}{1 - \phi^2} \\
cov(x_t, x_{t-h}) &= \phi^h \sigma_w^2 \frac{1 - \phi^{2(t-h)}}{1 - \phi^2}.
\end{aligned}$$

The result follows since

$$
\begin{aligned}
corr(x_t, x_{t-h}) &= \frac{cov(x_t, x_{t-h})}{\sqrt{var(x_t)}\sqrt{var(x_{t-h})}} \\
&= \phi^h \frac{\sqrt{var(x_{t-h})}\sqrt{var(x_{t-h})}}{\sqrt{var(x_t)}\sqrt{var(x_{t-h})}} \\
&= \phi^h \sqrt{\frac{var(x_{t-h})}{var(x_t)}}.
\end{aligned}
$$

(C) Since $\phi^2 \leq |\phi| < 1$ it follows that, for large t, $(\phi^2)^t \approx 0$ so that

$$
var(x_t) = \sigma_w^2 \frac{1-\phi^{2t}}{1-\phi^2} \approx \frac{\sigma_w^2}{1-\phi^2}.
$$

Also, for large t with respect to h,

$$
\begin{aligned}
corr(x_t, x_{t-h}) &= \phi^h \sqrt{\frac{var(x_{t-h})}{var(x_t)}} \\
&= \phi^h \sqrt{\frac{1-\phi^{2(t-h)}}{1-\phi^{2t})}} \\
&\approx \phi^h.
\end{aligned}
$$

Since for large t the mean function does not depend on time t and the autocovariance function depends on s and t (almost) only through their difference, one could say that x_t is "asymptotically stationary".

(D) Remember that if y_t is a stationary AR(1) process, $\mu_y(t) = 0$ and $\gamma_y(h) = \phi^h$. One could simulate $m + n$ iid observations from N(0,1), multiply them by σ_w, and plug them into the equations for x_t in the problem. From part **(C)** we know that, for large t, the mean and autocovariance functions of x_t are approximately equal to those of y_t so that, for large m, one could consider the last n simulated observations from x_t as coming from y_t.

(E) Again, note that

$$
\begin{aligned}
x_1 &= \frac{w_1}{\sqrt{1-\phi^2}} \\
x_2 &= \phi x_1 + w_2 = \phi \frac{w_1}{\sqrt{1-\phi^2}} + w_2 \\
x_3 &= \phi x_2 + w_3 = \phi(\phi \frac{w_1}{\sqrt{1-\phi^2}} + w_2) + w_3 = \frac{\phi^2 w_1}{\sqrt{1-\phi^2}} + \phi w_2 + w_3 \\
&\vdots
\end{aligned}
$$

so that, for $t = 1, 2, \ldots$ and $h \geq 0$,

$$
\begin{aligned}
x_t &= \frac{\phi^{t-1} w_1}{\sqrt{1-\phi^2}} + \sum_{i=0}^{t-2} \phi^i w_{t-i} \\
\mu_x &= \frac{\phi^{t-1} E[w_1]}{\sqrt{1-\phi^2}} + \sum_{i=0}^{t-2} \phi^i E[w_{t-i}] = 0 \\
cov(x_t, x_{t-h}) &= E[\left(\frac{\phi^{t-1} w_1}{\sqrt{1-\phi^2}} + \sum_{i=0}^{t-2} \phi^i w_{t-i} \right) \left(\frac{\phi^{t-h-1} w_1}{\sqrt{1-\phi^2}} + \sum_{j=0}^{t-h-2} \phi^j w_{t-h-j} \right)] \\
&= \sigma_w^2 \frac{\phi^{t-1}\phi^{t-h-1}}{1-\phi^2} + \sum_{i=0}^{t-2} \sum_{j=0}^{t-h-2} \phi^i \phi^j E[w_{t-h-j} w_{t-i}] \\
&= \sigma_w^2 \frac{\phi^{2t-h-2}}{1-\phi^2} + \sigma_w^2 \frac{\phi^h (1 - \phi^{2(t-h-1)})}{1-\phi^2} \\
&= \frac{\sigma_w^2}{1-\phi^2} (\phi^{2t-h-2} + \phi^h - \phi^h \phi^{2t-2h-2}) \\
&= \frac{\sigma_w^2 \phi^h}{1-\phi^2}
\end{aligned}
$$

using results from part **(A)**. The process is stationary since $\gamma_x(s,t)$ depends on s and t only through their difference and μ_x does not depend on t.