STATS 531 / ECON 677 WINTER 09
HOMEWORK 4

PROBLEM 3.3

(A) Rewriting the model as

$$\begin{aligned}
(1-0.8B+0.15B^2)x_t &= (1-0.3B)w_t \\
(1-0.3B)(1-0.5B)x_t &= (1-0.3B)w_t \\
(1-0.5B)x_t &= w_t,
\end{aligned}$$

we notice that there are redundancies in the original ARMA(1,1) model for x_t, which can be reduced to an AR(1) process with polynomials

$$\begin{aligned}
\phi(z) &= 1-0.5z \\
\theta(z) &= 1.
\end{aligned}$$

Letting $z^\star_{f,i}$ be the i^{th} root of the polynomial f, $z^\star_{\phi,1}=2$ and $\theta(z)$ has no roots. The process is causal because $|z^\star_{\phi,1}|>1$, i.e. x_t can be written as a one-sided linear process. Moreover, this linear process is unique because x_t is invertible, since for $|z|\leq 1$, $\theta(z)\neq 0$.

(B) Rewriting the model as

$$(1-B+0.5B^2)x_t = (1-B)w_t,$$

we see that there are no redundancies in the original ARMA(1,1) model for x_t since the polynomials

$$\begin{aligned}
\phi(z) &= 1-z+0.5z^2 \\
\theta(z) &= 1-z,
\end{aligned}$$

do not have factors in common. The roots are

$$\begin{aligned}
z^\star_\phi &= \frac{1\pm\sqrt{1^2-4(1)(0.5)}}{2(0.5)} == 1\pm\sqrt{-1}=1\pm i \\
z^\star_\theta &= 1
\end{aligned}$$

Since for $a,b\in\mathbb{R}$, $|a+bi|=\sqrt{a^2+b^2}$, the process is causal because $|z^\star_{\phi,1}|=|1+i|=|z^\star_{\phi,2}|=|1-i|=\sqrt{2}>1$, i.e. x_t can be written as a one-sided linear process. However, this linear process is not unique, since x_t is not invertible because $|z^{\star}_{\theta,1}| = 1$, which is not outside the unit circle.

PROBLEM 3.19 All three plots are consistent with the hypothesis that data are a realization of Gaussian white noise with $\sigma_w^2 = 1$, which we know is the case because the ARMA(1,1) model used to generate the data is redundant and can be reduced to that. An example of output of fitting an ARMA(1,1) to the simulated data in the plot is

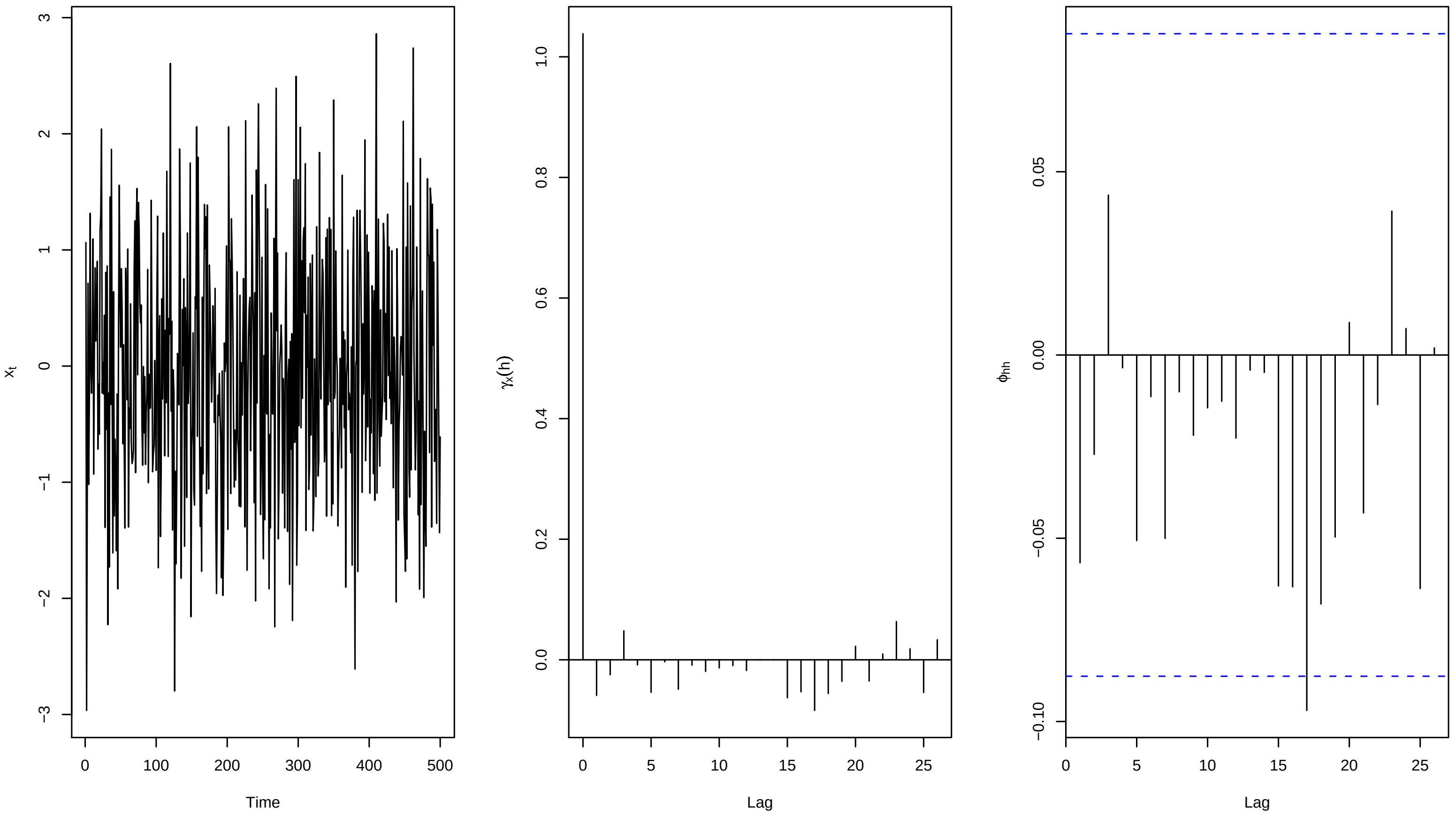
```
>ar.par <- list(order=c(1,0,1),ar=c(.9),ma=c(-.9))
>xt <- arima.sim(ar.par,500)

>arma11 <- arima(xt,order=c(1,0,1))
Warning message: possible convergence problem: optim gave code=1 in:
arima(xt, order = c(1, 0, 1))

> arma11
Call: arima(x = xt, order = c(1, 0, 1))
Coefficients:
          ar1     ma1  intercept
      -0.2773  0.2482    -0.0381
s.e.   2.1445  2.1515     0.0459

sigma^2 estimated as 1.102: loglikelihood = -733.72,  aic = 1475.45.
```

Note that the optimization function issued a warning and that $\hat{\phi} \approx -\hat{\theta}$. An explanation for this is as follows. Since the arima command fits the data by numerically maximizing the likelihood, any combination of ARMA(1,1) parameters that generates data that “resembles” Gaussian white noise with $\sigma_w^2 = 1$ should have a large likelihood. In particular, any combination of the form $\phi = -\theta$ is in fact a Gaussian white noise model and has the same likelihood. Hence, one should not be surprised if the optimization algorithm returns a warning given this feature of the likelihood surface. Models with $\hat{\phi} \approx -\hat{\theta}$ might have a slightly greater likelihood and one should expect the MLEs to be close in magnitude and of opposite sign. Note also that σ_w^2 is estimated precisely, as one would expect with 500 observations. For these particular simulated data and numerical optimization run, the coefficients appear to be not statistically significant, given the large standard errors. Note however that it is possible that for other simulated data, and/or other fits to the same simulated data, the standard errors may be small enough to suggest significance.



x_t
Time
−3
−2
−1
0
1
2
3
0
100
200
300
400
500
$\gamma_x(h)$
Lag
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
ϕ_{hh}
Lag
−0.10
−0.05
0.00
0.05
0
5
10
15
20
25