STATS 531 / ECON 677 WINTER 09
HOMEWORK 5

PROBLEM 4.3 Let $\cos_k(t) = \cos(2\pi\omega_k t)$ and similarly for $\sin_k(t)$.

$$\begin{aligned}
cov(x_t, x_{t+h}) &= E[x_t x_{t+h}] - E[x_t]E[x_{t+h}] \\
&= E\Big[\sum_{j=1}^{q}[U_{1j}\cos_j(t) + U_{2j}\sin_j(t)]\sum_{k=1}^{q}[U_{1k}\cos_k(t+h) + \qquad (1) \\
&\quad U_{2k}\sin_k(t+h)]\Big] \\
&= E\Big[\sum_{j=1}^{q}\sum_{k=1}^{q}\big\{U_{1j}\cos_j(t)U_{1k}\cos_k(t+h) + U_{2j}\sin_j(t)U_{1k}\cos_k(t+h) + \\
&\quad U_{1j}\cos_j(t)U_{2k}\sin_k(t+h) + U_{2j}\sin_j(t)U_{2k}\sin_k(t+h)\big\}\Big] \\
&= \sum_{k=1}^{q} E[U_{1k}]^2 cos_k(t)\cos_k(t+h) + E[U_{2k}]^2\sin_k(t)\sin_k(t+h) \qquad (2) \\
&= \sum_{k=1}^{q}\sigma_k^2\Big(cos_k(t)\cos_k(t+h) + \sin_k(t)\sin_k(t+h)\Big) \\
&= \sum_{k=1}^{q}\sigma_k^2\cos_k(h), \qquad (3)
\end{aligned}$$

where (1) follows because $U_{..}$ have mean zero, (2) follows by independence and (3) follows by the trigonometric identity in the book.

PROBLEM 4.7 First note

$$\begin{aligned}
cov(z_t, z_{t+h}) &= E[z_t z_{t+h}] - E[z_t]E[z_{t+h}] \\
&= E[x_t y_t x_{t+h} y_{t+h}] \qquad (4) \\
&= E[x_t x_{t+h}]E[y_t y_{t+h}] \qquad (5) \\
&= \gamma_x(h)\gamma_y(h), \qquad (6)
\end{aligned}$$

where (4) follows since $E[z_t] = E[x_t]E[y_t] = 0$ by independence and (5) follows by independence for all t and $t+h$. Now, by definition

$$
\begin{aligned}
f_z(\omega) &= \sum_{h=-\infty}^{h=\infty} \gamma_z(h) e^{-i2\pi\omega h} \\
&= \sum_{h=-\infty}^{h=\infty} \gamma_x(h)\gamma_y(h) e^{-i2\pi\omega h} && (7) \\
&= \sum_{h=-\infty}^{h=\infty} \gamma_x(h) \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} f_y(\nu) e^{i2\pi\nu h} d\nu \right] e^{-i2\pi\omega h} && (8) \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\sum_{h=-\infty}^{h=\infty} \gamma_x(h) f_y(\nu) e^{i2\pi\nu h} e^{-i2\pi\omega h} \right] d\nu && (9) \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} f_y(\nu) \left[\sum_{h=-\infty}^{h=\infty} \gamma_x(h) e^{-i2\pi(\omega-\nu)h} \right] d\nu \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} f_y(\nu) f_x(\omega-\nu) d\nu, && (10)
\end{aligned}
$$

where (7) follows by (6), (8) by definition, in (9) we assume that we can interchange the order of summation and integration, and (10) follows by definition. One could also have used the result in the class notes that the Fourier transform swaps product and convolution. By definition,

$$
\begin{aligned}
f_z(\omega) &= F\big(\gamma_z(h)\big)(\omega) \\
&= F\big(\gamma_x(h)\gamma_y(h)\big)(\omega) && (11) \\
&= \Big(F\big(\gamma_x(h)\big) * F\big(\gamma_y(h)\big)\Big)(\omega) && (12) \\
&= \big(f_x * f_y\big)(\omega) && (13) \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} f_y(\nu) f_x(\omega-\nu) d\nu, && (14)
\end{aligned}
$$

where (11) follows by (6), (12) by the "swapping property" from class, (13) by definition and (14) by the definition of convolution (since $-\frac{1}{2} \leq \omega \leq \frac{1}{2}$ by definition).