STATS 531 / ECON 677 WINTER 09
HOMEWORK 7

PROBLEM 1

(i) Let I be the identity matrix of appropriate dimensions, $\mathbf{0}$ be a matrix of zeros of appropriate dimensions and A' be the transpose of A. Then,

$$\begin{aligned}
cov(\boldsymbol{z},\boldsymbol{y}) &= cov(\boldsymbol{x}-\Sigma_{xy}\Sigma_y^{-1}\boldsymbol{y},\boldsymbol{y}) \\
&= cov(\boldsymbol{x},\boldsymbol{y}) - cov(\Sigma_{xy}\Sigma_y^{-1}\boldsymbol{y},\boldsymbol{y}) \\
&= \Sigma_{xy} - \Sigma_{xy}\Sigma_y^{-1}cov(\boldsymbol{y},\boldsymbol{y})I' = \Sigma_{xy} - \Sigma_{xy}\Sigma_y^{-1}\Sigma_y \\
&= \Sigma_{xy} - \Sigma_{xy} = \mathbf{0}.
\end{aligned}$$

Hence $\boldsymbol{z}$ and $\boldsymbol{y}$ are uncorrelated in the sense that the correlation coefficient $\rho_{ij} = \frac{cov(z_i,y_j)}{\sigma_{z_i}\sigma_{y_j}}$ for $\forall i,j$ will be zero. Since 2 uncorrelated Gaussian random variables are also independent, one may argue that $\boldsymbol{z}$ and $\boldsymbol{y}$ are independent as well. Formally, letting $\boldsymbol{w} = (\boldsymbol{z},\boldsymbol{y})'$,this follows because

$$\begin{aligned}
\frac{1}{\sqrt{(2\pi)^n|\Sigma_z|}}e^{-\frac{1}{2}(\boldsymbol{z}-\mu_z)'\Sigma_z^{-1}(\boldsymbol{z}-\mu_z)} &= f_z(\boldsymbol{z}) \\
&= f_{z|y}(\boldsymbol{z}|\boldsymbol{y}),
\end{aligned}$$

which may be shown using $cov(\boldsymbol{z},\boldsymbol{y}) = \mathbf{0}$ and the properties of the block matrices to show that $|\Sigma_w| = |\Sigma_z\Sigma_y - \mathbf{0}| = |\Sigma_z||\Sigma_y|$ and $\Sigma_w^{-1} = \begin{pmatrix} \Sigma_z^{-1} & \mathbf{0} \\ \mathbf{0} & \Sigma_y^{-1} \end{pmatrix}$.

(ii)

$$\begin{aligned}
E[\boldsymbol{x}|\boldsymbol{y}] &= E[\boldsymbol{z}+\Sigma_{xy}\Sigma_y^{-1}\boldsymbol{y}|\boldsymbol{y}] \\
&= E[\boldsymbol{z}|\boldsymbol{y}] + \Sigma_{xy}\Sigma_y^{-1}\boldsymbol{y} \\
&= E[\boldsymbol{z}] + \Sigma_{xy}\Sigma_y^{-1}\boldsymbol{y} \\
&= \mu_x - \Sigma_{xy}\Sigma_y^{-1}\mu_y + \Sigma_{xy}\Sigma_y^{-1}\boldsymbol{y} = \mu_x + \Sigma_{xy}\Sigma_y^{-1}(\boldsymbol{y}-\mu_y)
\end{aligned}$$

where the first equality follows by the definition of $\boldsymbol{z}$ and the third one by independence of $\boldsymbol{z}$ and $\boldsymbol{y}$.

(iii)

$$
\begin{aligned}
V[\boldsymbol{x}|\boldsymbol{y}] &= V[\boldsymbol{z} + \Sigma_{xy}\Sigma_y^{-1}\boldsymbol{y}|\boldsymbol{y}] \\
&= V[\boldsymbol{z}|\boldsymbol{y}] \\
&= V[\boldsymbol{z}] \\
&= V[\boldsymbol{x} - \Sigma_{xy}\Sigma_y^{-1}\boldsymbol{y}] \\
&= V[\boldsymbol{x}] - cov[\Sigma_{xy}\Sigma_y^{-1}\boldsymbol{y}, \boldsymbol{x}] - cov[\boldsymbol{x}, \Sigma_{xy}\Sigma_y^{-1}\boldsymbol{y}] + V[\Sigma_{xy}\Sigma_y^{-1}\boldsymbol{y}] \\
&= \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{xy}I' - I\Sigma_{xy}(\Sigma_{xy}\Sigma_y^{-1})' + \Sigma_{xy}\Sigma_y^{-1}\Sigma_y(\Sigma_{xy}\Sigma_y^{-1})' \\
&= \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{xy}
\end{aligned}
$$

PROBLEM 2 z_1 and z_2 are linear combinations of the complex normal variables x and y. Hence, $\boldsymbol{z}$ is bivariate complex normal with mean μ_z and covariance matrix Σ_z. Since $A = [a_{ij}]$,

$$
\boldsymbol{z} = A \begin{pmatrix} x \\ y \end{pmatrix}
$$

and $E[\boldsymbol{z}] = \boldsymbol{0}$ because x and y have mean zero. By definition, the complex normal random variables x and y may be written as

$$
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ s \end{pmatrix} + i \begin{pmatrix} v \\ t \end{pmatrix}
$$

where u,v,s and t are independent $N(0, \sigma^2/2)$. Letting $A = B + iC$ (as in the hint),

$$
\begin{aligned}
\boldsymbol{z} &= (B + iC)\left[\begin{pmatrix} u \\ s \end{pmatrix} + i \begin{pmatrix} v \\ t \end{pmatrix}\right] \\
&= B\begin{pmatrix} u \\ s \end{pmatrix} - C\begin{pmatrix} v \\ t \end{pmatrix} + i\left[C\begin{pmatrix} u \\ s \end{pmatrix} + B\begin{pmatrix} v \\ t \end{pmatrix}\right] = \boldsymbol{d} + i\boldsymbol{e}.
\end{aligned}
$$

From class, if $V\left[\begin{pmatrix} \boldsymbol{d} \\ \boldsymbol{e} \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} R & -S \\ S & R \end{pmatrix}$, and $S' = -S$, then $V[\boldsymbol{z}] = R + iS$. Note that

$$\begin{aligned}
V[\boldsymbol{d}] &= V\left[B\begin{pmatrix} u \\ s \end{pmatrix} - C\begin{pmatrix} v \\ t \end{pmatrix}\right] \\
&= V\left[B\begin{pmatrix} u \\ s \end{pmatrix}\right] + V\left[C\begin{pmatrix} v \\ t \end{pmatrix}\right] - Bcov\left[\begin{pmatrix} u \\ s \end{pmatrix}, \begin{pmatrix} v \\ t \end{pmatrix}\right]C' - Ccov\left[\begin{pmatrix} v \\ t \end{pmatrix}, \begin{pmatrix} u \\ s \end{pmatrix}\right]B' \\
&= \frac{\sigma^2}{2}BIB' + \frac{\sigma^2}{2}CIC' = \frac{\sigma^2}{2}(BB' + CC') = V[\boldsymbol{e}].
\end{aligned}$$

where the last equality follows by independence of u,v,s and t. Also note that

$$\begin{aligned}
cov[\boldsymbol{d}, \boldsymbol{e}] &= cov\left[B\begin{pmatrix} u \\ s \end{pmatrix} - C\begin{pmatrix} v \\ t \end{pmatrix}, C\begin{pmatrix} u \\ s \end{pmatrix} + B\begin{pmatrix} v \\ t \end{pmatrix}\right] \\
&= E\left[\Big(B(u,s)' - C(v,t)'\Big)\Big((u,s)C' + (v,t)B'\Big)\right] \\
&= E\left[B(u,s)'(u,s)C' + B(u,s)'(v,t)B' - C(v,t)'(u,s)C' - C(v,t)'(v,t)B'\right] \\
&= E\left[B\begin{pmatrix} u^2 & us \\ us & s^2 \end{pmatrix}C' + B\begin{pmatrix} uv & ut \\ sv & st \end{pmatrix}B' - C\begin{pmatrix} vu & vs \\ tu & ts \end{pmatrix}C' - C\begin{pmatrix} v^2 & vt \\ vt & t^2 \end{pmatrix}B'\right] \\
&= \frac{\sigma^2}{2}BIC' - \frac{\sigma^2}{2}CIB' = \frac{\sigma^2}{2}(BC' - CB') = -cov[\boldsymbol{e}, \boldsymbol{d}].
\end{aligned}$$

Finally, letting $R = 2V[\boldsymbol{d}]$ and $S = 2cov[\boldsymbol{e}, \boldsymbol{d}]$, it follows that

$$\begin{aligned}
\Sigma_z &= R + iS \\
&= \sigma^2(BB' + CC') + i\sigma^2(CB' - BC'), \\
&= \sigma^2(BB' + CC' - i(BC' - CB'))
\end{aligned}$$

and, by definition,

$$\begin{aligned}
A\bar{A}' &= (B + iC)(\overline{B + iC})' = (B + iC)(B - iC)' \\
&= BB' - iBC' + iCB' - (-1)CC' \\
&= BB' + CC' - i(BC' - CB'),
\end{aligned}$$

so that $\boldsymbol{z} \sim N^c(\boldsymbol{0}, \sigma^2 A\bar{A}')$