STATS 531 / ECON 677 WINTER 09
HOMEWORK 9

Remember the multiplication rule that $p(x, y, z) = p(x|y, z)p(y|z)p(z)$, which may be proved by repeatedly using of the definition of conditional density, and let $y_{1:t} = (y_1, \ldots, y_t)$. The prediction recursion is

$$\begin{aligned}
p(x_t|y_{1:t-1}) &= \int_{\mathcal{X}} p(x_t, x_{t-1}|y_{1:t-1})dx_{t-1} \\
&= \int_{\mathcal{X}} \frac{p(x_t, x_{t-1}, y_{1:t-1})}{p(y_{1:t-1})}dx_{t-1} \\
&= \int_{\mathcal{X}} \frac{p(x_t|x_{t-1}, y_{1:t-1})p(x_{t-1}|y_{1:t-1})p(y_{1:t-1})}{p(y_{1:t-1})}dx_{t-1} \\
&= \int_{\mathcal{X}} p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})dx_{t-1}
\end{aligned}$$

where the equalities follow respectively by the definition of marginal density, the definition of conditional density, the multiplication rule and by the property of state space models that x_t and $y_{1:t-1}$ are independent given x_{t-1}. The filtering recursion is

$$\begin{aligned}
p(x_t|y_{1:t}) &= \frac{p(x_t, y_{1:t-1}, y_t)}{p(y_{1:t-1}, y_t)} \\
&= \frac{p(y_t|x_t, y_{1:t-1})p(x_t|y_{1:t-1})p(y_{1:t-1})}{p(y_t|y_{1:t-1})p(y_{1:t-1})} \\
&= \frac{p(y_t|x_t)p(x_t|y_{1:t-1})}{p(y_t|y_{1:t-1})}
\end{aligned}$$

where the equalities follow respectively by the definition of conditional density, the multiplication rule and the property of state space models that y_t and $y_{1:t-1}$ are independent given x_t. The last recursion is, for $1 < t < T$,

$$\begin{aligned}
p(x_t|y_{1:T}) &= \frac{p(x_t, y_{1:t-1}, y_{t:T})}{p(y_{1:t-1}, y_{t:T})} \\
&= \frac{p(y_{t:T}|x_t, y_{1:t-1})p(x_t|y_{1:t-1})p(y_{1:t-1})}{p(y_{t:T}|y_{1:t-1})p(y_{1:t-1})} \\
&= \frac{p(y_{t:T}|x_t)p(x_t|y_{1:t-1})}{p(y_{t:T}|y_{1:t-1})} \propto p(y_{t:T}|x_t)p(x_t|y_{1:t-1})
\end{aligned}$$

where the equalities follow by the definition of conditional density, the multiplication rule and the property of state space models that $y_{t:T}$ and $y_{1:t-1}$ are independent given x_t. The proportionality follows because $p(y_{t:T}|y_{1:t-1})$ is not a function of x_t.