Stat 620: Sample Final Exam Solutions

November 27, 2007

1(a) Consider entries into state 1 as renewals with a reward of one at each entry. Using the renewal reward theorem, long run proportion of entries into state 1 is given by

$$\begin{aligned}\lim_{n\to\infty}\frac{R_n}{n} &= \frac{E[\text{reward in 1 cycle}]}{E[\text{length of 1 cycle}]}\\ &= \frac{1}{2(1/2)+3(1/2)} = 0.4\end{aligned}$$

(b) Similarly to (a) the long run proportion of entries into 2 is

$$\frac{1}{2(1/2)+3(1/2)} = 0.4$$

Thus $\pi_1 = \pi_2 = 0.4$, $\pi_3 = 0.2$ are the limiting transition probabilities. The limiting state probabilities are given by

$$P_i = \frac{\pi_i \mu_i}{\sum_i \pi_i \mu_i}$$

Thus $P_1 = 2/9$, $P_2 = 4/9$ and $P_3 = 1/3$.

2 Let $\{\pi_i\}$ be the stationary probabilities for the original chain, so (using reversibility) $\pi_i q_{ij} = \pi_j q_{ji}$. Now, look for a solution $\{\pi_i^*\}$ to the equation

$$\pi_i^* q_{ij}^* = \pi_j^* q_{ji}^* \tag{1}$$

Then, for $i \in A$, $j \notin A$

$$\pi_i^* q_{ij}^* = c\pi_i^* q_{ij} = \pi_j^* q_{ji}$$

Set

$$\pi_i^* = \begin{cases} a\pi_i & \text{if } i \in A \\ b\pi_i & \text{if } i \notin A \end{cases}$$

and require $b/a = c$ and $a\sum_{i\in A}\pi_i + b\sum_{i\notin A}\pi_i = 1$. It can be shown that $\{\pi_i^*\}$ satisfy (1) as follows. If $i \in A$, $j \in A$

$$\pi_i^* q_{ij}^* = a\pi_i q_{ij} = a\pi_j q_{ji} = \pi_j^* q_{ji}^*$$

If $i \in A$, $j \notin A$

$$\pi_i^* q_{ij}^* = a\pi_i c q_{ij} = b\pi_j q_{ji} = \pi_j^* q_{ji}^*$$

If $i \notin A$, $j \notin A$

$$\pi_i^* q_{ij}^* = b\pi_i q_{ij} = b\pi_j q_{ji} = \pi_j^* q_{ji}^*$$

Thus, $\{\pi_i^*\}$ is a probability distribution which solves the detailed balance equations, implying both that q_{ij}^* is reversible and π_i^* are stationary probabilities.

3(a) Let $X_i(t)$ be 1 if the ith component is working at time t and 0 otherwise. Let $X(t) = (X_1(t), \cdots, X_n(t))$. Then $X(t)$ is a continuous time Markov chain, with state space $\{0,1\}^n$ and transition rates given by

$$q\left((x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n), (x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n)\right) = \lambda_i$$

$$q\left((x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n), (x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n)\right) = \mu_i \alpha^{\sum_j (1-x_j)}$$

with other transition rates being zero.

(b) The detailed balance equations are

$$\begin{aligned} & \pi(x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n) q\left((x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n), (x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n)\right) \\ = \; & \pi(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n) q\left((x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n), (x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n)\right) \end{aligned}$$

i.e.

$$\frac{\pi(x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n)}{\pi(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n)} = \frac{\mu_i}{\lambda_i} \alpha^{\sum_j (1-x_j)}$$

Now set

$$\pi(x) = \frac{\prod_{i=1}^n (\mu_i/\lambda_i)^{x_i} \alpha^{f(x)}}{\sum_{y \in \{0,1\}^n} \prod_{i=1}^n (\mu_i/\lambda_i)^{y_i} \alpha^{f(y)}} \tag{2}$$

Then $\pi(x) \geq 0$ and $\sum_{x \in \{0,1\}^n} \pi(x) = 1$. The detailed balance equations are satisfied if

$$\textstyle\sum_j (1 - x_j) = f((x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n)) - f((x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n)).$$

This is solved by

$$\begin{aligned} f(x) &= n + (n-1) + \cdots + (n - \textstyle\sum_j x_j + 1) \\ &= n \textstyle\sum_j x_j - \sum_j x_j (\sum_j x_j - 1)/2. \end{aligned}$$

Therefore, the chain is reversible and (2) gives the stationary probabilities. Since the chain is irreducible, these are also the limiting probabilities.

4

$$E[B(T)] = E[2 - 4T] = 2 - 4E[T]$$

Assume, we can apply the stopping theorem to the martingale $B(t)$. Now, $E[T] < \infty$ and $E[|B(t+s) - B(t)| \big| B(u), 0 \leq u \leq t]$ depends on s but not on t, so conditions analogous to (iii) for the discrete stopping theorem apply. Then $E[B(T)] = E[B(0)] = 0$. So $E[T] = 1/2$.

5

$$\begin{aligned} Cov(B(t), B(s) - sB(t)/t) &= Cov(B(t), B(s)) - (s/t)Cov(B(t), B(t)) \\ &= s - (s/t)t = 0 \end{aligned}$$

Applying the hint gives independence.

6 $Z(t)$ is a Gaussian process, since $(Z(t_1), \cdots, Z(t_n))$ is a linear combination of the multivariate Gaussian random variable

$$\left(B(\frac{t_1}{1-t_1}), \cdots, B(\frac{t_n}{1-t_n})\right)$$

Now

$$E[Z(t)] = E[(1-t)B(\frac{t}{1-t})] = (1-t)E[B(\frac{t}{1-t})] = 0 \tag{3}$$

Also for $0 \le s \le t \le 1$, $s/(1-s) \le t/(1-t)$ and

$$\begin{aligned} Cov(Z(s), Z(t)) &= (1-s)(1-t)Cov\left(B(\frac{s}{1-s}), B(\frac{t}{1-t})\right) \\ &= (1-s)(1-t)\frac{s}{1-s} = s - st \end{aligned} \tag{4}$$

Since (3) and (4) match the mean and covariance function of a Brownian bridge, the result is shown.

7 Let $f(x) = \sqrt{x}$. Then, from the transformation rule

$$\begin{aligned} \mathrm{d}Y(t) &= f'(X(t))\mathrm{d}X(t) + (1/2)f''(X(t))\sigma_X^2(X(t))\mathrm{d}t \\ &= (1/2)X(t)^{-1/2}[(bX(t)+c)\mathrm{d}t + \sqrt{4X(t)}\mathrm{d}B(t)] \\ &\quad +(1/2)(-1/4)X(t)^{-3/2}4X(t)\mathrm{d}t \\ &= \frac{1}{2}\left(bY(t) + \frac{c-1}{Y(t)}\right)\mathrm{d}t + \mathrm{d}B(t) \end{aligned}$$

8 Let $P_A = P[S_T = A]$. Note that T is a stopping time, $S_{min(T,n)}$ is bounded and S_n is a martingale. Thus the stopping theorem gives $E[S_T] = E[S_0] = 0$. So, $AP_A - B(1-P_A) = 0$, which gives

$$P_A = \frac{B}{A+B}$$

Now, $S_n^2 - n$ is also a martingale, and since the increments are bounded and $E[T] < \infty$, the stopping theorem gives

$$E[S_T^2 - T] = E[S_0^2 - 0] = 0$$

So $A^2P_A + B^2P_B = E[T]$. Thus

$$E[T] = \frac{A^2B}{A+B} + \frac{B^2A}{A+B} = AB$$

9

$$\begin{aligned} P[R_n = R_{n-1}+1] &= P[S_{n-1} \neq S_n, S_{n-2} \neq S_n, \cdots, S_0 \neq S_n] \\ &= P[X_n \neq 0, X_{n-1}+X_n \neq 0, \cdots, X_1+\cdots+X_n \neq 0] \\ &= P[X_1 \neq 0, X_1+X_2 \neq 0, \cdots, X_1+\cdots+X_n \neq 0] \text{ (Duality)} \\ &= P[S_1 \neq 0, S_2 \neq 0, \cdots, S_n \neq 0] \end{aligned}$$

Setting $p_n = P[R_n = R_{n-1}+1]$

$$\begin{aligned} E[R_n] &= E[1+\sum_{n=1}^{\infty} \mathbf{I}(R_n = R_{n-1}+1)] \\ &= 1+\sum_{n=1}^{\infty} P[R_n = R_{n-1}+1] \\ &= 1+\sum_{n=1}^{\infty} p_n \end{aligned}$$

Now, p_n is a decreasing non-negative sequence and thus has a limit. Hence

$$\lim_{n\to\infty} \frac{E[R_n]}{n} = \lim_{n\to\infty} p_n = P[S_k \neq 0 \text{ for all } k \geq 1]$$

10 $\{X_n\}$ is a martingale since $E[|X_n|] < \infty$ and

$$\begin{aligned} E[X_{n+1}|X_1, \cdots, X_n] &= X_n(\alpha X_n + 1 - \alpha) + (1-X_n)\alpha X_n \\ &= X_n \end{aligned}$$

Since $0 \leq X_n \leq 1$, the Martingale Convergence theorem says that X_n converges almost surely. The only two possible convergence points are 0 and 1. Thus X_n converges to a Bernoulli random variable. Also, $E[X_n] = E[X_0] = 1/2$.