

Stat 620 sample final exam questions

1. Consider a system that can be in either state 1, 2 or 3. Each time the system enters state i it remains there for a random amount of time having mean μ_i and then makes a transition into state j with probability P_{ij} . Suppose $P_{12} = 1$, $P_{21} = P_{23} = 1/2$, $P_{31} = 1$.
 - (a) What proportion of the transitions take the system into state 1?
 - (b) If $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 3$, then what proportion of the time does the system spend in each state?
2. Consider a time reversible continuous-time Markov chain having infinitesimal transition rates q_{ij} and limiting probabilities $\{P_i\}$. Let A denote a set of states for this chain, and consider a new continuous-time Markov chain with transition rates q_{ij}^* given by

$$q_{ij}^* = \begin{cases} cq_{ij} & \text{if } i \in A, j \notin A \\ q_{ij} & \text{otherwise} \end{cases}$$

where c is an arbitrary positive number. Show that this chain remains time reversible and find its limiting probabilities.

3. Consider a system of n components such that the working times of component i , $i = 1, \dots, n$, are exponentially distributed with rate λ_i . When failed, however, the repair rate of component i depends on how many other components are down. Specifically, suppose that the instantaneous repair rate of component i when there are a total of k failed components is $\alpha^k \mu_i$.
 - (a) Explain how we can analyze this system as a continuous-time Markov chain. Define the states and give the parameters of the chain.
 - (b) Show that, in steady state, the chain is time reversible and compute the limiting probabilities.
4. Let $T = \min\{t : B(t) = 2 - 4t\}$, i.e., T is the first time that standard Brownian motion hits the line $2 - 4t$. Use the martingale stopping theorem to find $E[T]$.
5. For $s < t$, argue that $B(s) - \frac{s}{t}B(t)$ and $B(t)$ are independent.

Hint: Gaussian random variables are independent if and only if they are uncorrelated. In general, lack of correlation does not imply independence.
6. If $B(t)$ is standard Brownian motion and $Z(t) = (1 - t)W(\frac{t}{1-t})$, show that $Z(t)$ is a Brownian bridge.
7. If $X(t)$ is a diffusion process taking positive real values, with infinitesimal parameters $\mu(x, t) = bx + c$ with $c > 0$ and $\sigma^2(x, t) = 4x$, what are the infinitesimal parameters of the diffusion process $Y(t) = \sqrt{X(t)}$?

8. Let S_n be the simple random walk, which is equally likely to increase or decrease one unit at each time point. Suppose $S_0 = 0$, and let $T = \min\{n : S_n = A \text{ or } S_n = -B\}$ for positive integers A and B . Use a martingale argument to find $E[T]$.

Hint: the martingale $E[\exp\{\theta S_n\}]$ used for the random walk with drift does not work here, as the only solution to $E[\exp\{\theta S_n\}] = 1$ is $\theta = 0$. S_n itself is a martingale, but you will need another martingale, possibly one involving S_n^2 .

9. Let $\{X_n, n \geq 1\}$ be independent, identically distributed random variables taking integer values. Let $S_0 = 0$ and $S_n = \sum_{i=1}^n S_i$. The *range* R_n of S_0, S_1, \dots, S_n is the number of distinct values taken by the sequence. Use duality to show that

$$P(R_n = R_{n-1} + 1) = P(S_1 \neq 0, \dots, S_n \neq 0).$$

Hence argue that, as $n \rightarrow \infty$,

$$\frac{1}{n}E[R_n] \rightarrow P(S_k \neq 0 \text{ for all } k \geq 1).$$

10. Let $\{X_n, n \geq 0\}$ be a Markov process for which X_0 is uniform on $(0, 1)$ and, conditional on X_n ,

$$X_{n+1} = \begin{cases} \alpha X_n + 1 - \alpha & \text{with probability } X_n \\ \alpha X_n & \text{with probability } 1 - X_n \end{cases}$$

where $0 < \alpha < 1$. Discuss the limiting properties of the sequences $\{X_n, n \geq 1\}$ and $\{E[X_n], n \geq 1\}$.