Stat 620: Final 2006 Solutions

(1)

$$\begin{aligned} P(X_{n+1}=j|X_n=i) &= P(X_{n+1}=j|X_n=i,Y_{n+1}=j)P(Y_{n+1}=j|X_n=i) \\ &= q_{ij}\frac{a_j}{a_i+a_j} \end{aligned}$$

$$\begin{aligned} P(X_{n+1}=i|X_n=i) &= P(X_{n+1}=i|X_n=i,Y_{n+1}=j)P(Y_{n+1}=j|X_n=i) \\ &= q_{ij}\frac{a_i}{a_i+a_j} \end{aligned}$$

So,

$$\begin{aligned} P_{ij} &= q_{ij}\frac{a_j}{a_i+a_j} \\ P_{ii} &= q_{ij}\frac{a_i}{a_i+a_j} \end{aligned} \tag{1}$$

Also,

$$\begin{aligned} P_{ji} &= P(X_{n+1}=i|X_n=j) \\ &= P(X_{n+1}=i|X_n=j,Y_{n+1}=i)P(Y_{n+1}=i|X_n=j) \\ &= q_{ji}\frac{a_i}{a_i+a_j} \end{aligned}$$

Note that by the symmetry of the algorithm $q_{ij}=q_{ji}$. The limiting probabilities can be verified using the detailed balance equations.

$$\begin{aligned} \pi_i P_{ij} &= \pi_i q_{ij}\frac{a_j}{a_i+a_j} \\ &= \frac{a_i}{\sum_i a_i} q_{ij}\frac{a_j}{a_i+a_j} \end{aligned}$$

$$\begin{aligned} \pi_j P_{ji} &= \pi_j q_{ji}\frac{a_i}{a_i+a_j} \\ &= \frac{a_j}{\sum_j a_j} q_{ji}\frac{a_i}{a_i+a_j} \end{aligned}$$

Thus, the detailed balance equation is satisfied since $q_{ij}=q_{ji}$. We can also check that $\sum_i \pi_i = 1$. Thus π_is are the limiting probabilities.

(2.a) This is a birth-death process for the taxis, with birth rate $\lambda = 1$ and death rate $\mu = 2$. Now, using the balance equation,

$$\begin{aligned} P_n\lambda &= P_{n+1}\mu \\ \sum_{n=0}^{\infty} P_n &= 1 \end{aligned}$$

After some algebra,

$$P_k = \frac{1}{2^{k+1}}$$

Therefore, expected no. of taxis waiting=$\sum_{k=0}^{\infty} kP_k = 1$

(2.b) by PASTA, proportion of customers who get taxi
=P(no. of taxis waiting is more than 1)
=$1 - P_0$ =1/2

(3) Use a martingle approach similar to question 6.10 (Ross) from Homework 9. The answer is $E[N] = p^{-3}q^{-3} + p^{-2}q^{-2} + p^{-1}q^{-1}$ where $q = 1 - p$.

(4) Define $\max_{t_1 \le s \le t_2} B(s) = B_T$. Now,

$$\begin{aligned} P(B_T \ge x | B(t_1) = a) &= P(\max_{t_1 \le s \le t_2} (B(s) - B(t_1)) \ge x - a | B(t_1) = a) \\ &= P(\max_{0 \le s \le t_2 - t_1} (B(s) \ge x - a)) \\ &= P(T_{x-a} \le t_2 - t_1) \\ &= 2P(B(t_2 - t_1) \ge x - a) \\ &= 2(1 - \Phi(\frac{x-a}{\sqrt{t_2 - t_1}})) \\ &= 2\Phi(\frac{a-x}{\sqrt{t_2 - t_1}}) \end{aligned}$$

We note that,

$$P(B_T \ge x | B(t_1) = a) = 1 \text{ if } a \ge x$$

Thus,

$$\begin{aligned} P(B_T \ge x) &= E[P(B_T \ge x | B(t_1))] \\ &= \frac{1}{\sqrt{2\pi t_1}} \int_{-\infty}^{x} 2(1 - \Phi(\frac{x-a}{\sqrt{t_2 - t_1}}))e^{-a^2/2t_1}\, da + P(B(t_1) > x) \\ &= \int_{-\infty}^{x} 2(1 - \Phi(\frac{x-a}{\sqrt{t_2 - t_1}}))\phi(\frac{a}{\sqrt{t_1}})\, da + \Phi(-\frac{x}{\sqrt{t_1}}) \end{aligned}$$

(5.i) A diffusion is a continuous time, continuous state process with continuous sample paths that possesses the Markov property. An O-U process is a diffusion and $f(x) = e^x$ is a continuous function. Transforming the sample paths by a continuous function preserves the continuity of the sample paths. The transformation also does not affect the Markov property, so the transformed process is also a diffusion.

(5.ii) We have

$$X(t) = \log Y(t)$$

Now,using the transformation formula with $f(x) = e^x$,

$$\begin{aligned} \mu_Y(y,t) &= \mu_X(x,t)f'(x) + \frac{1}{2}\sigma_X^2(x,t)f"(x) \\ &= -\alpha x e^x + \frac{1}{2}e^x \\ &= -\alpha y \log y + \frac{1}{2}y \end{aligned}$$

and

$$\begin{aligned} \sigma_Y^2(y,t) &= \sigma_X^2(x,t)[f'(x)]^2 \\ &= [e^x]^2 \\ &= y^2 \end{aligned}$$