

Statistics 620, Fall 2007

Name: _____ UMID #: _____

Final Exam

- There are 5 questions, each worth 10 points.
- You are allowed a calculator and a single-sided sheet of notes.
- Credit will be given for clear explanation and justification, as well as for getting the correct answer.
- Cross out any working that you do not wish to be considered as part of your solution. You are advised not to erase unfinished working since partial credit may be available for an indication that an appropriate method was attempted, even if it was later rejected.

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

1. Suppose that traffic on a road follows a Poisson process with rate λ cars per minute. A chicken needs a gap of length at least c minutes in the traffic to cross the road. To compute the time the chicken will have to wait to cross the road, let t_1, t_2, t_3, \dots be the interarrival times for the cars and let $J = \min\{j : t_j > c\}$. If $T_n = t_1 + \dots + t_n$, then the chicken will start to cross the road at time T_{J-1} and complete his journey at time $T_{J-1} + c$.

(a) [4 points]. Suppose T is exponentially distributed with rate λ . Find $E[T|T < c]$.

Hint: Using the identity $ET = P(T < c)E[T|T < c] + P(T > c)E[T|T > c]$ leads to a nice solution, though you can also solve the problem by direct calculation.

(b) [6 points] Use part (a) to show $E(T_{J-1} + c) = (e^{\lambda c} - 1)/\lambda$. If you have not solved (a), you may leave your answer in terms of $E[T|T < c]$.

2. We investigate a martingale solution to the same situation problem from Question 1. As before, traffic on a road follows a Poisson process with rate λ cars per minute. A chicken needs a gap of length at least c minutes in the traffic to cross the road. t_1, t_2, t_3, \dots are the interarrival times for the cars and $J = \min\{j : t_j > c\}$. If $T_n = t_1 + \dots + t_n$, then the chicken will start to cross the road at time T_{J-1} and complete his journey at time $T_{J-1} + c$. Note that $T_n - (n/\lambda)$ is a martingale.

(a) [3 points] Argue that J is a stopping time for t_1, t_2, \dots , and explain why $J - 1$ is not a stopping time.

(b) [7 points] Use a martingale argument to show that $E(T_{J-1} + c) = (e^{\lambda c} - 1)/\lambda$.

3. We study a queue with impatient customers. Customers arrive at a single server as a Poisson process with rate λ and require an exponential amount of service with rate μ . customers waiting in line are impatient and if they are not in service they will leave at rate δ independent of their position in the queue. Show that for any $\delta > 0$ the system has a stationary distribution, and find an expression for this distribution.

4. A cocaine dealer is standing on a street corner. Customers arrive at times of a Poisson process with rate λ . The customer and the dealer then disappear from the street for an amount of time with distribution G while the transaction is completed. Customers that arrive during this time go away never to return.

(a) [5 points] At what rate does the dealer make sales? Explain your reasoning.

(b) [5 points] What fraction of customers are lost? Explain your reasoning.

5. Let $B(t)$ be standard Brownian motion and let $Z(t) = B(t) - tB(1)$ be a Brownian bridge. Define $X(t) = (1+t)Z(t/(1+t))$. Show that $X(t)$ is a standard Brownian motion.

Hint: recall that $Z(t)$ is a Gaussian diffusion with $E[Z(t)] = 0$ and $Cov(Z(s), Z(t)) = s \wedge t - st$.