1(a).

$$\begin{aligned}\mathbb{E}(T|T<c) &= \frac{\mathbb{E}(T;T<c)}{\mathbb{P}(T<c)} \\ &= \frac{\int_0^c t\lambda e^{-\lambda t}dt}{1-e^{-c}} = \frac{-ce^{-\lambda c}+\frac{1}{\lambda}(1-e^{-\lambda c})}{1-e^{-c}}\,.\end{aligned}$$

1(b). Note that

$$T_{J-1}+c = c\,\mathbf{1}_{\{t_1>c\}} + (t_1+c+\sum_{k=2}^{J-1}t_k)\mathbf{1}_{\{t_1\le c\}}\,.$$

Taking expectation on both sides above, we have

$$\mathbb{E}(T_{J-1}+c) = c\,\mathbb{P}(t_1>c)+\mathbb{P}(t_1\le c)\,\mathbb{E}\,(t_1|t_1\le c)+\mathbb{P}(t_1\le c)\,\mathbb{E}\left((\sum_{k=2}^{J-1}t_k+c)\middle|\,t_1\le c\right)\,. \quad (1)$$

Note that in above, by result in (a), the second term on the r.h.s. of (1) equals $-ce^{-\lambda c}+\frac{1}{\lambda}(1-e^{-\lambda c})$, and the third term equals $\mathbb{P}(t_1\le c)\,\mathbb{E}(T_{J-1}+c)$ by memoriless property. Hence, it follows that

$$\begin{aligned}\mathbb{E}(T_{J-1}+c) &= \frac{ce^{-\lambda c}+\left(-ce^{-\lambda c}+\frac{1}{\lambda}(1-e^{-\lambda c})\right)}{\mathbb{P}(t_1>c)} \\ &= \frac{\frac{1}{\lambda}\left(1-e^{-\lambda c}\right)}{e^{-\lambda c}} = \frac{1}{\lambda}\left(e^{\lambda c}-1\right)\,.\end{aligned}$$

2(a). Let $X_n = T_n - \frac{n}{\lambda}$. J is a stopping time because

$$\begin{aligned}\{J=n\} &= \{T_1<c, T_2-T_1<c,\ldots,T_{n-1}-T_{n-2}<c, T_n-T_{n-1}>c\} \\ &= \{X_1+\frac{1}{\lambda}<c, X_2-X_1+\frac{1}{\lambda}<c\ldots,X_{n-1}-X_{n-2}+\frac{1}{\lambda}<cX_n-X_{n-1}>c\}\,,\end{aligned}$$

which is determined by the value of $X_1,\ldots,X_n$. Similarly, $\{J-1=n\}=\{J=n+1\}$ is determined by the value of $X_1,\ldots,X_{n+1}$. Thus, J is a stopping time, but $J-1$ is not.

2(b). First, we have

$$\mathbb{E}(T_{J-1}+c) = \mathbb{E}(T_J-t_J+c) = \mathbb{E}(T_J-\frac{J}{\lambda}+\frac{J}{\lambda}-t_J+c) = \mathbb{E}(T_J-\frac{J}{\lambda})+\frac{\mathbb{E}J}{\lambda}-\mathbb{E}(t_J)+c\,.$$

We calculate the three expectations in the r.h.s. above respectively. It is easy to see that J has geometric distributuion with parameter $p=\mathbb{P}(t_1>c)=e^{-\lambda c}$. Then $\mathbb{E}J=e^{c\lambda}$. Also,

$$\mathbb{E}(t_J) = \mathbb{E}(t_1|t_1>c) = c+\mathbb{E}(t_1) = c+\frac{1}{\lambda}.$$

To calculate $\mathbb{E}(T_J-\frac{J}{\lambda})$, since we have shown that $X_n=T_n-\frac{n}{\lambda}$ is a martingale and J is a stopping time, by the fact that $\mathbb{E}J<\infty$ and that

$$\mathbb{E}\left(|X_{n+1}-X_n|\,\middle|X_1,\ldots,X_n\right) = \mathbb{E}\left(\left|t_1-\frac{1}{\lambda}\right|\,\middle|X_1,\ldots,X_n\right) \le \mathbb{E}(t_1)+\frac{1}{\lambda}<\infty\,,$$

we apply the martingale stopping theorem to obtain

$$\mathbb{E}(T_J-\frac{J}{\lambda}) = \mathbb{E}(T_1-\frac{1}{\lambda}) = 0\,.$$

Plugging in, we have

$$\mathbb{E}(T_{J-1}+c) = 0 + \frac{e^{\lambda c}}{\lambda} - c - \frac{1}{\lambda} + c = \frac{e^{\lambda c}-1}{\lambda}\,.$$

3. Let $X(t)$ denote the process. It has states $\{0,1,2,\dots\}$. We can write the transition matrix as follows.

$$\begin{aligned} q_{i,i+1} &= \lambda \\ q_{i,i-1} &= \mu + (i-1)\delta, \quad i \geq 1 \\ q_{i,k} &= 0 \text{otherwise} \end{aligned}$$

It is clear that we have a death and birth process. THen, the stationary distributuion P_i satisfies:

$$\sum_{i=0}^{\infty} P_i = 1 \quad \text{and} \quad P_i q_{i,i+1} = P_{i+1} q_{i+1,i}\,. \tag{2}$$

By solving (2), we have

$$P_i = \frac{\lambda^i}{\prod_{j=0}^{i-1}(\mu + j\delta)} P_0\,, \quad P_0 = \left(1 + \sum_{i=1}^{\infty} \frac{\lambda^i}{\prod_{j=0}^{i-1}(\mu + j\delta)}\right)^{-1}\,.$$

To see the existence of stationary distribution, one can check that $P_0 > 0$. Indeed, since $\lambda > 0$, there exists $J \in \mathbb{N}$ such that $\mu + J\delta > \lambda$. Then, it follows that

$$\sum_{i=1}^{\infty} \frac{\lambda^i}{\prod_{j=0}^{i-1}(\mu + j\delta)} < \sum_{i=1}^{J-1} \frac{\lambda^i}{\prod_{j=0}^{i-1}(\mu + j\delta)} + \sum_{i=J}^{\infty} \left(\frac{\lambda}{\mu + J\delta}\right)^i < \infty\,.$$

4(a). We can define a renewal process, each cycle ending when a transaction is finished. Let X_i denote the length of each cycle. Let T be a random variable with distributuion G. Then,

$$\mathbb{E}(X_i) = \frac{1}{\lambda} + \mathbb{E}(T)\,.$$

The, the percentage of time on transaction equals $\frac{\mathbb{E}(T)}{\frac{1}{\lambda} + \mathbb{E}(T)}$.

4(b). The fraction of customers that are lost is the stationary distribution of state 1, i.e., $\frac{\mathbb{E}(T)}{\frac{1}{\lambda} + \mathbb{E}(T)}$.

5. Since $Z(t)$ is a Brownian bridge, it is a Gaussian diffusion. Then $X(t)$ is a Gaussian diffusion. Now we prove

$$\mathbb{E}(X(t)) = 0 \quad \text{and} \quad \mathrm{Cov}(X(s), X(t)) = s \wedge t - st\,.$$

First,

$$\begin{aligned} \mathbb{E}(X(t)) &= (1+t)\,\mathbb{E}\left(Z\left(\frac{t}{1+t}\right)\right) = (1+t)\,\mathbb{E}\left(B\left(\frac{t}{1+t}\right) - \frac{t}{1+t}B(1)\right) \\ &= (1+t)\left(0 - \frac{t}{1+t}\right) = 0\,. \end{aligned}$$

Next, by the fact that $Z(t)$ is a Brownian bridge,

$$\begin{aligned} \mathrm{Cov}(X(s), X(t)) &= \mathrm{Cov}\left((1+s)Z\left(\frac{s}{1+s}\right), (1+t)Z\left(\frac{t}{1+t}\right)\right) \\ &= (1+s)(1+t)\frac{s}{1+s} \wedge \frac{t}{1+t} - \frac{st}{(1+s)(1+t)} \\ &= s(1+t) \wedge t(1+s) - st = s \wedge t - st\,. \end{aligned}$$

We have thus proved that $X(t)$ is a standard Brownian motion.