Statistics 620, Fall 2008

Name: ______________________ **UMID #:** ______________________

Final Exam

- **There are 5 questions, each worth 10 points.**
- **You are allowed a calculator and a single-sided sheet of notes.**
- **Credit will be given for clear explanation and justification, as well as for getting the correct answer.**
- **Cross out any working that you do not wish to be considered as part of your solution. You are advised not to erase unfinished working since partial credit may be available for an indication that an appropriate method was attempted, even if it was later rejected.**

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

1. A bag contains four blue balls and five red balls. George plays a game in which he bets $1 to draw a ball at random from the bag. If he draws a red ball he wins $1; otherwise he loses $1. Assume that the balls are drawn with replacement and that George starts with $50 in the hope of reaching $100 at which point he stops playing. However, if he loses all his money before this, the game also ends and he becomes bankrupt. What is the probability that the game ends with George being $50 richer than he was at the begininning?

2. Let $N(t)$ be a Poisson process with rate 1 and let $\{S_n\}$ be the corresponding sequence of arrival times. For a fixed time T, let $U = S_{N(T)}$.

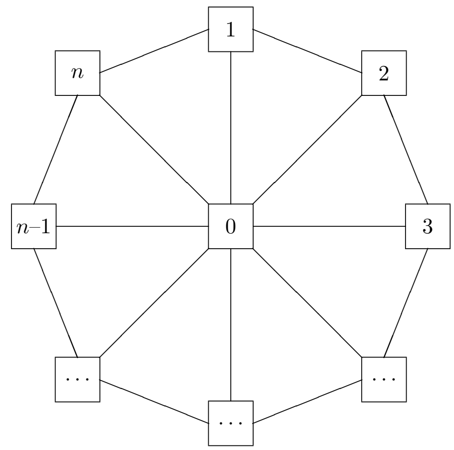
(a) [3 points] Is U a stopping time for $N(t)$? Answer yes or no, and briefly explain. You should define what you mean by a stopping time for a continuous-time process; in class we defined this only for discrete time, but the concept extends readily to continuous time.

(b) [2 points] Is it the case that, conditional on $U = u$, $N(T) - N(U) \sim Poisson(T - u)$? Answer yes or no, and explain briefly.

(c) [5 points] Find $E[T - U]$. Hint: one approach is to condition on $N(T)$; you may use, without proof, standard results on the distribution of order statistics of arrival times given $N(T)$.

3. This question concerns a simplified version of the method used by the Google search engine to rank the importance of web sites. Imagine a graph in which nodes are web pages and edges are hyperlinks (i.e., connections between web pages). Now imagine a web surfer who carries out a random walk on the web by picking a hyperlink at random from the current page. The limiting proportion of the time that the random walk spends at node i is a measure of how central i is to the web, and therefore how highly Google should rank the page. Find the limiting probabilities for a random walk on the graph shown, which is a wheel with nodes $1, \ldots, n$ at the end of n spokes and a node 0 at the center.

Note: here we consider an undirected graph, which is easier to analyze. Of course, web hyperlinks are in fact directional.



4. Let $Y(t) = \int_0^t B(u)\, du$, where $B(t)$ is standard Brownian motion. Evaluate the following:

(a) [3 points] $E[Y(t)]$.

(b) [3 points] The variance of $Y(t)$.

(c) [2 points] The covariance of $Y(t)$ and $B(t)$.

(d) [2 points] The conditional distribution of $Y(t)$ given that $B(t) = x$.

5. Consider a continuous-time Markov chain $X(t)$ taking values on the integers and having transition rates q_{ij}. Suppose $X(0) = 0$. Let A denote a set of states that does not include 0 and set T to be the first hitting time of A, i.e., $T = \min\{t > 0 : X(t) \in A\}$. Suppose that T is finite with probability one. Set $Q_i = \sum_{j \in A} q_{ij}$ and consider the random variable $H = \int_0^T Q_{X(t)}\, dt$.

(a) [7 points] Find the hazard rate function of H. That is, find

$$h(s) = \lim_{\delta \to 0} P\{s < H < s + \delta \mid H > s\}/\delta.$$

Hint: one approach is to condition on the state of the chain at the random time τ_s for which $\int_0^{\tau_s} Q_{X(t)}\, dt = s$.

(b) [3 points] What is the distribution of H?