

Statistics 620 Midterm Sample Solutions

1. Let $N(t)$ count the number of fish caught by time t , and let $\{X_i, i = 1, 2, \dots\}$ be a sequence of random variables giving the weight of each fish. We assume that these weights are iid and independent of $N(t)$. Let $N = N(2)$ count the fish caught in 2 hours. Let $W = \sum_{i=1}^N X_i$. Then,

$$E[W] = EE[W|N] = E[N]E[X_1]$$

$$\begin{aligned} \text{Var}(W) &= E[\text{Var}(W|N)] + \text{Var}(E[W|N]) \\ &= E[N \text{Var}(X_1)] + \text{Var}(N E[X_1]) \\ &= E[N] \text{Var}(X_1) + (E[X_1])^2 \text{Var}(N). \end{aligned}$$

Here, $E[N] = 120 \times 0.1 = 12$ and $\text{Var}(N) = E[N] = 12$. So, $E[W] = 12 \times 4 = 48$ pounds, $sd(W) = \sqrt{12 \times 4 + 12 \times 4^2} = 15.5$ pounds.

2. Let $N_1(t)$ count burnout events, and $N_2(t)$ count handyman replacements. Then $N_3(t) = N_1(t) + N_2(t)$ is a Poisson process with rate $\frac{1}{200} + \frac{3}{365} = 0.0132 \text{ day}^{-1}$ (assuming independence of $N_1(t)$ and $N_2(t)$).

(a) Average time between replacements $= (0.0132)^{-1} = 75.6$ days.

(b)

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{N_1(t)}{N_3(t)} &= \lim_{t \rightarrow \infty} \frac{N_1(t)}{t} / \lim_{t \rightarrow \infty} \frac{N_3(t)}{t} \\ &= (1/200)/0.0132 = 0.38 \text{ w.p.1} \end{aligned}$$

3. This is in the notes.

4. Let X_n denote the ordering of the cards after the n^{th} operation. X_n is a Markov Chain whose state space is orderings of $1, \dots, 52$. Note that X_n is irreducible (since in 52 operations we can reach any order) and aperiodic (since $\mathbf{P}[X_n = X_{n-1}] > 0$). Then X_n has a unique stationary distribution which is also the limiting distribution. Let π be the uniform distribution on orderings, so $\pi_i = 1/52!$. Note that the transition matrix of X_n is

$$P_{ij} = \begin{cases} 1/52 & \text{if there is a card that can be placed on top to get } j \text{ from } i \\ 0 & \text{else} \end{cases}$$

Then $\sum_i \pi_i P_{ij} = \sum_{i: P_{ij} > 0} \frac{1}{52!} \times \frac{1}{52} = 52 \times \frac{1}{52!} \times \frac{1}{52} = \frac{1}{52!} = \pi_j$. This gives the uniform distribution as the limiting distribution.