Sample midterm questions (solutions posted online)

1. Levin catches trout according to a Poisson process with rate 0.1 minute^{-1}. Suppose that the trout weigh an average of 4 pounds with a standard deviation of 2 pounds. Find the mean and standard deviation of the total weight of fish he catches in two hours.

2. A light bulb has a lifetime that is exponential with a mean of 200 days. When it burns out, a janitor replaces it immediately. In addition, there is a handyman who comes, on average, three times per year according to a Poisson process and replaces the lightbulb as "preventative maintenance."

   (a) What is the average time between bulb replacements?

   (b) In the long run, what fraction of the replacements are due to failure?

3. Let $\{N(t), t \geq 0\}$ be a renewal process, with corresponding arrival times $\{S_n\}$ and inter-arrival times $X_n = S_n - S_{n-1}$ with $E[X_n] = \mu$. Show that $lim_{t\to\infty} N(t)/t = 1/\mu$, with probability one.

   Hints: (i) it may be helpful to notice that $S_{N(t)} \leq t \leq S_{N(t)+1}$. (ii) you are asked to show what amounts to a strong law of large numbers for $N(t)$; you can use without proof the strong law of large numbers for S_n.

4. Consider the following approach to shuffling a deck of n cards. Starting with any initial ordering of the cards, one of the numbers $1, 2, \ldots, 52$ is chosen at random and with equal probability. If number i is chosen, we move the card from position i in the deck to the top, i.e. to position 1. We repeatedly perform the same operation. Show that, in the limit, the deck is perfectly shuffled in the sense that the resultant ordering is equally likely to be any of the $n!$ possible orderings.