Stat 620: Exam 1 Solutions

October 23, 2006

1 Let $N_1(t)$ be the number of accidents not resulting in a lawsuit by time t and $N_2(t)$ be the number of accidents resulting in a lawsuit by time t. Then $N_1(t)$ is a Poisson process with rate $\lambda(1-p)$, $N_2(t)$ is a Poisson process with rate λp and $N_1(t)$ is independent of $N_2(t)$.

$$\begin{aligned} P(A=n|T=t) &= P(N_1(t)+N_2(t)=n|N_2(t)=1, N_2(s)=0 \text{ for } s<t) \\ &= P(N_1(t)=n-1),\ N_1 \text{ and } N_2 \text{ are independent} \\ &= \frac{e^{-\lambda(1-p)t}(\lambda(1-p)t)^{n-1}}{(n-1)!} \end{aligned} \tag{1}$$

2(i)

$$\begin{aligned} E[A(t)] &= E[t-S_{N(t)}] \\ &= \int_0^t (t-u)\mathrm{d}F_{S_{N(t)}}(u)\ \text{, since } 0 \le S_{N(t)} < t \\ &= (t-0)(\overline{F}(t)(1+\mathrm{d}m(0)) + \int_0^t (t-u)\overline{F}(t-u)\mathrm{d}m(u) \\ &= t\overline{F}(t) + \int_0^t (t-u)\overline{F}(t-u)\mathrm{d}m(u)\ \text{, since } \mathrm{d}m(0)=0 \end{aligned}$$

(ii) Assuming X has finite second moment, $E[X^2] = \int_0^\infty 2x\overline{F}(x)\mathrm{d}x$, as proved in a homework problem. Also this implies that

$$\lim_{t\to\infty} t\overline{F}(t) = 0$$

Thus

$$\begin{aligned}
\lim_{t\to\infty} E[A(t)] &= \lim_{t\to\infty}[t\overline{F}(t) + \int_0^t (t-u)\overline{F}(t-u)\mathrm{d}m(u)] \\
&= 0 + \lim_{t\to\infty}\int_0^t (t-u)\overline{F}(t-u)\mathrm{d} \\
&= \frac{1}{E[X]}\int_0^t t\overline{F}(t)\mathrm{d}t \text{ , using Key Renewal Theorem} \\
&= \frac{E[X^2]}{2E[X]}
\end{aligned}$$

3(i)

$$P[X_{n+1}=x|X_n=1,X_{n-1},\cdots] = P[X_{n+1}=x|X_n='?F'] = \begin{cases} P[F|X_n='?F'] = .5 & x=1 \\ P[S|X_n='?F'] = .5 & x=2 \\ 0 & x=3 \end{cases}$$

$$P[X_{n+1}=x|X_n=2,X_{n-1},\cdots] = P[X_{n+1}=x|X_n='FS'] = \begin{cases} P[F|X_n='FS'] = .5 & x=1 \\ P[S|X_n='FS'] = .5 & x=3 \\ 0 & x=2 \end{cases}$$

$$P[X_{n+1}=x|X_n=3,X_{n-1},\cdots] = P[X_{n+1}=x|X_n='SS'] = \begin{cases} P[F|X_n='SS'] = .2 & x=1 \\ P[S|X_n='SS'] = .8 & x=3 \\ 0 & x=2 \end{cases}$$

The above shows that $\{X_n\}$ is a markov chain and also gives the transition matrix as

$$P = \begin{pmatrix} .5 & .5 & 0 \\ .5 & 0 & .5 \\ .2 & 0 & .8 \end{pmatrix}$$

(ii) The chain is irreducible since all states communicate with each other and it is aperiodic since $P_{11} > 0$. Therefore the chain is ergodic and has a unique stationary distribution satisfying $\pi = \pi P$. The set of linear equations is easy to solve, and gives $\pi_1 = 4/11$, $\pi_2 = 2/11$ and $\pi_3 = 5/11$.

$$\begin{aligned}
\lim_{n\to\infty} P[S] &= \lim_{n\to\infty}\sum_x P[S|X_n=x]P[X_n=x] \\
&= \sum_x P[S|X_n=x]\lim_{n\to\infty} P[X_n=x] \\
&= \sum_x P[S|X_n=x]\pi_x \\
&= .5\pi_1 + .5\pi_2 + .8\pi_3 = 7/11
\end{aligned}$$