Statistics 620
Fall, 2007

Name: ______________________________ **UMID #:** ______________________________

Midterm Exam

- **There are 3 questions, each worth 10 points.**
- **You are allowed a calculator and a single-sided sheet of notes.**
- **Credit will be given for clear explanation and justification, as well as for getting the correct answer.**
- **Cross out any working that you do not wish to be considered as part of your solution. You are advised not to erase unfinished working since partial credit may be available for an indication that an appropriate method was attempted, even if it was later rejected.**

Problem	Points	Your Score
1	10	
2	10	
3	10	
Total	30	

1. Suppose that the number of hours between successive arrivals of the train at a station is uniformly distributed on $(0, 1)$. Passengers arrive according to a Poisson process with a rate λ. Suppose a train has just left the station. Let X denote the number of people who get on the next train.

(a) [5 points] Find $E[X]$.

(b) [5 points] Find $Var(X)$.

2. Suppose that customers arrive at a single-server system in accordance with a Poisson process with rate λ. Upon arriving a customer must pass through a door that leads to the server. However, each time someone passes through, the door becomes locked for the next t units of time. An arrival finding the door locked is lost, and a cost c is incurred by the system. An arrival finding the door unlocked passes through to the server. If the server is free, the customer enters service; if the server is busy, the customer departs without service and a cost K is incurred. The service time of a customer is exponential with rate μ.

(a) [2 points] Explain why times at which a customer arrives to find the door unlocked are regeneration times for this system.

From part (a), the cost incurred between successive regeneration times is a renewal-reward process. We now proceed to compute the long-run cost per unit time.

(b) [2 points] Defining a new cycle to begin each time a customer arrives to find the door unlocked, calculate the expected length of a cycle.

(c) [2 points] Let C_1 denote the cost incurred during a cycle due to arrivals finding the door locked. Calculate $E[C_1]$.

(d) [2 points] Let C_2 denote the cost incurred during a cycle due to an arrival finding the door unlocked but the server busy. Calculate $E[C_2]$.

(e) [2 points] Combining the answers from parts (a)-(d), give an expression for the long run cost per unit time.

3. This question concerns a periodic Markov chain. Recall that a state i is periodic with period d if $P_{ii}^n = 0$ whenever n is not divisible by d, and that periodicity is a class property. Suppose $\{W_n\}$ is an irreducible positive recurrent Markov chain with period 2, with W_n taking values in a countable state space S.

For parts (b) and (c), you may assume the existence of the set A from part (a). Therefore you do not have to complete (a) to proceed with (b) and (c).

(a) [4 points] Show that there exists a subset A of S such that $P(W_1 \in A | W_0 = i)$ is 0 if $i \in A$ and 1 if $i \notin A$.

Hint: one possibility is to show that the following gives a construction of such a set A: pick some arbitrary state j and let $A = \{k : P_{jk}^{2n} > 0 \text{ for some } n = 0, 1, 2, \ldots\}$.

(b) [3 points] Let $V_n^{(1)} = W_{2n}$, with $V_0^{(1)}$ taking a value in A. Show that $\{V_n^{(1)}\}$ is an ergodic Markov chain with state space A. (We say $\{V_n^{(1)}\}$ is $\{W_{2n}\}$ restricted to A.)

(c) [3 points] Let π be the stationary distribution of $\{W_n\}$. Define $\pi^{(1)}$ to be the stationary distribution of $V_n^{(1)}$ and $\pi^{(2)}$ to be the stationary distribution of $\{V_n^{(2)}\}$, which is $\{W_{2n}\}$ restricted to the state space A^c. Describe the relationship between these three distributions.

Hint: You may adopt the convention that $\pi^{(1)}$ and $\pi^{(2)}$ are defined everywhere in S, with $\pi^{(1)}(i) = 0$ if $i \notin A$ and $\pi^{(2)}(i) = 0$ if $i \notin A^c$. You may wish to show that $\pi^{(1)} = \pi^{(2)}P$, where P is the transition matrix for $\{W_n\}$.