Statistics 620
Fall, 2007

1. Suppose that the number of hours between successive arrivals of the train at a station is uniformly distributed on $(0, 1)$. Passengers arrive according to a Poisson process with a rate λ. Suppose a train has just left the station. Let X denote the number of people who get on the next train. Find

(a) $E[X]$

Solution: Let T be the time until the next train. Then $T \sim$ Uniform $(0, 1)$. Conditioning on T, we have that $E[X] = E[E[X|T]]$. Now $E[X|T = t] = \lambda t$. So $E[X] = \lambda E[T] = \lambda/2$.

(b) $Var(X)$

Solution:

$$\begin{aligned} Var(X) &= E[Var(X|T)] + Var(E[X|T]) \\ &= E[\lambda T] + Var(\lambda T) \\ &= \lambda E[T] + \lambda^2 Var(T) \\ &= \lambda(1/2) + \lambda^2(1/12). \end{aligned}$$

2. Suppose that customers arrive at a single-server system in accordance with a Poisson process with rate λ. Upon arriving a customer must pass through a door that leads to the server. However, each time someone passes through, the door becomes locked for the next t units of time. An arrival finding the door locked is lost, and a cost c is incurred by the system. An arrival finding the door unlocked passes through to the server. If the server is free, the customer enters service; if the server is busy, the customer departs without service and a cost K is incurred. The service time of a customer is exponential with rate μ.

(a) [2 points] Explain why times at which a customer arrives to find the door unlocked are regeneration times for this system.

Solution: Directly after the new customer arrives, there will always be one customer being served (either the new arrival, or the customer previously being served). By the memoryless property of the service times, the time to the next service is independent of which of these two has occurred. Also, by the memoryless property of the inter-arrival times, the future arrivals form a Poisson process independent of the event of the arrival of the new customer.

From part (a), the cost incurred between successive regeneration times is a renewal-reward process. We now proceed to compute the long-run cost per unit time.

(b) [2 points] Defining a new cycle to begin each time a customer arrives to find the door unlocked, calculate the expected length of a cycle.

Solution: The start of a cycle means that a customer arrived to find the door unlocked. So we know that t time units will pass while the door remains closed. Upon opening, we wait for the arrival of a new customer, regardless of whether the server is free or not. This will take $(1/\lambda)$ time in expectation, due to the memoryless property. So

$$E[\text{length of a cycle}] = t + 1/\lambda.$$

(c) [2 points] Let C_1 denote the cost incurred during a cycle due to arrivals finding the door locked. Calculate $E[C_1]$.

Solution: A cost of c units is incurred for each customer that arrives during the first t time units of a cycle. Hence, $E[C_1] = \lambda tc$.

(d) [2 points] Let C_2 denote the cost incurred during a cycle due to an arrival finding the door unlocked but the server busy. Calculate $E[C_2]$.

Solution: There will be either one or zero instance of cost K per cycle, depending on whether the first arrival after the opening of the door occurs after or before the completion of the service time. With probability $e^{-\mu t}$ the current customer has a service time greater than t time units. In this case, we use the memoryless property to "reset" the system. Conditional on this event, the occurrence of a cost K event is equivalent to an Exponential(λ) random variable being less than an independent Exponential(μ) variable, which has probability $\lambda/(\lambda+\mu)$. Therefore, $E[C_2] = K\, e^{-\mu t}\, \lambda/(\lambda+\mu)$.

(e) [2 points] Combining the answers from parts (a)-(d), give an expression for the long run cost per unit time.

Solution: The average cost will be given by $lim_{t\to\infty} R(t)/t$, where $R(t)$ is the cost incurred by time t. This limit is equal to $E[R]/E[X]$, where $E[R] = \lambda tc + Ke^{-\mu t}\lambda/(\lambda+\mu)$ and $E[X] = t + (1/\lambda)$.

3. This question concerns a periodic Markov chain. Recall that a state i is periodic with period d if $P^n_{ii} = 0$ whenever n is not divisible by d, and that periodicity is a class property. Suppose $\{W_n\}$ is an irreducible positive recurrent Markov chain with period 2, with W_n taking values in a countable state space S.

For parts (b) and (c), you may assume the existence of the set A from part (a). Therefore you do not have to complete (a) to proceed with (b) and (c).

(a) [4 points] Show that there exists a subset A of S such that $P(W_1 \in A | W_0 = i)$ is 0 if $i \in A$ and 1 if $i \notin A$.

Hint: one possibility is to show that the following gives a construction of such a set A: pick some arbitrary state j and let $A = \{k : P^{2n}_{jk} > 0 \text{ for some } n = 0, 1, 2, \ldots\}$.

Solution: Let A be as in the hint. Argue by contradiction: suppose there exist $i \in A$ and $k \in A$ with $P_{ik} > 0$. By construction, there are integers m and n such that $P^{2m}_{ji} > 0$ and $P^{2n}_{jk} > 0$. Also, the irreducibility of $\{W_n\}$ guarantees some ℓ with $P^{\ell}_{kj} > 0$. Thus, $P^{2m+1+\ell}_{jj} > P^{2m}_{ji} P_{ik} P^{\ell}_{kj} > 0$ and $P^{2n+\ell}_{jj} > P^{2n}_{jk} P^{\ell}_{kj} > 0$. One of $2m+1+\ell$ and $2n+\ell$ must be odd, so j cannot have period 2. This shows $P(W_1 \in A | W_0 = i)$ is 0 if $i \in A$. An equivalent argument shows $P(W_1 \in A^c | W_0 = i)$ is 0 if $i \in A^c$.

(b) [3 points] Let $V^{(1)}_n = W_{2n}$, with $V^{(1)}_0$ taking a value in A. Show that $\{V^{(1)}_n\}$ is an ergodic Markov chain with state space A. (We say $\{V^{(1)}_n\}$ is $\{W_{2n}\}$ restricted to A.)

Solution: We must argue that $\{V^{(1)}_n\}$ is aperiodic, irreducible and positive recurrent.

If $\{V^{(1)}_n\}$ has period d then $\{W_n\}$ has period $2d$ (you could check this) and so by assumption $\{V^{(1)}_n\}$ has period 1.

For any $i \in A$, there is some n with $P^n_{ij} > 0$ (by irreducibility of $\{W_n\}$), where j is the specific state in (a). Clearly n must be even, due to the periodicity of $\{W_n\}$. Thus j communicates with i for each $i \in A$. By the transitivity of the communication relationship, this shows that $\{V_n^{(1)}\}$ is irreducible.

Positive recurrence of $\{V_n^{(1)}\}$ is directly inherited from $\{W_n\}$.

(c) [3 points] Let π be the stationary distribution of $\{W_n\}$. Define $\pi^{(1)}$ to be the stationary distribution of $V_n^{(1)}$ and $\pi^{(2)}$ to be the stationary distribution of $\{V_n^{(2)}\}$, which is $\{W_{2n}\}$ restricted to the state space A^c. Describe the relationship between these three distributions.

Hint: You may adopt the convention that $\pi^{(1)}$ and $\pi^{(2)}$ are defined everywhere in S, with $\pi^{(1)}(i) = 0$ if $i \notin A$ and $\pi^{(2)}(i) = 0$ if $i \notin A^c$. You may wish to show that $\pi^{(1)} = pi^{(2)}P$, where P is the transition matrix for $\{W_n\}$.

Solution: Given the hint, together with the equivalent result $\pi^{(2)} = \pi^{(1)}P$, it is immediate that $\pi(i) = (\pi^{(1)}(i) + \pi^{(2)}(i))/2$ satisfies $\pi P = \pi$ and $\sum_i \pi(i) = 1$.

To show the hint, note that the limiting distribution for W_n at even times, plus one extra application of P, must give the limiting distribution at odd times.