1. Let $\pi_0 = (1,0)$ denote the initial distribution. Let π_n denote the distribution at time n. We have

$$\pi_n = \pi_0 P^n\,, \quad \text{with} \quad P = \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}.$$

There exists a matrix Ω such that

$$P = \Omega\Lambda\Omega^{-1} \quad , \text{ with} \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where λ_1 and λ_2 are the distinct eigenvalues of the matrix P. Hence,

$$P^n = \left(\Omega\Lambda\Omega^{-1}\right)^n = \Omega\Lambda^n\Omega^{-1} = \Omega \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \Omega^{-1}.$$

We want to calculate

$$\mathbb{P}(X_n = 1) = \pi_n(1) = a\lambda_1^n + b\lambda_2^n\,.$$

To find λ_1 and λ_2, solving the characteristic equation we obtain

$$\lambda^2 - 1.7\lambda + 0.7 = 0\,,$$

which gives $\lambda_1 = 1, \lambda_2 = 0.7$. It follows

$$\pi_n(1) = a + b0.7^n\,.$$

Note also that $\pi_0(1) = 1$ and $\pi_1(1) = 0.9$. It follows

$$\left\{\begin{array}{lcl} a+b & = & 1 \\ a+0.7b & = & 0.9 \end{array}\right. \Rightarrow \left\{\begin{array}{l} a = \frac{2}{3} \\ b = \frac{1}{3} \end{array}\right.$$

Finally,

$$\pi_n(1) = a + b0.7^n = \frac{2}{3} + \frac{1}{3} \times 0.7^n\,.$$

2. Define the process is *on* at t if $X_{N(t)+1} \le x$ and *off* if $X_{N(t)+1} > x$. Then, by the alternating renewal theorem,

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}(X_{N(t)+1} \le x) &= \lim \mathbb{P}(\text{the process is } on \text{ at time } t) \\ &= \frac{\mathbb{E}(on \text{ time in a interval})}{\exp(\text{time in a interval})} = \frac{\mathbb{E}(X\mathbf{1}_{X\le x})}{\mathbb{E}X} = \frac{\int_0^x y dF(y)}{\mathbb{E}(X)}\,. \end{aligned}$$

When $N(t)$ is a Poisson process with rate λ, we have

$$\begin{aligned} \int_0^x y dF(y) &= \int_0^x y\lambda\exp\{-\lambda y\}dy = -y\exp\{-\lambda y\}|_0^x + \int_0^x \exp\{-\lambda y\}dy \\ &= -x\exp\{-\lambda x\} + \frac{1}{\lambda}\left(1 - \exp\{-\lambda x\}\right)\,, \end{aligned}$$

and

$$\mathbb{E}(X) = \frac{1}{\lambda}\,.$$

Thus,

$$\lim_{t\to\infty} \mathbb{P}(X_{N(t)+1} \le x) = 1 - (1+\lambda x)\exp\{-\lambda x\}\,.$$

Note that this is different from the distribution of interval time of Poisson process with parameter λ, which equals $F(x) = 1 - \exp\{-\lambda x\}$.

3. (i) of B is immediate. To show (ii), it suffices to show $N(t)$ has stationary increments. That is, the distribution of $N(t+h) - N(t)$ depends only on h. This follows from (iii) of A. To complete the proof, we show that (iii) and (iv) of B hold by Taylor expansion:

$$\mathbb{P}(N(h)=1) = \mathbb{P}(N(h)-N(0)=1) = \frac{\exp\{-\lambda h\}\lambda h}{1!} = \lambda h(1-\lambda h + o(h)) = \lambda h + o(h)\,,$$

and

$$\begin{aligned}\mathbb{P}(N(h)\ge 2) &= 1 - \mathbb{P}(N(h)=1) - \mathbb{P}(N(h)=0) = 1 - (\lambda h + 1)\exp\{\lambda h\}\\ &= 1 - (1+\lambda h)(1-\lambda h + o(h)) = o(h)\,.\end{aligned}$$

4. (i) and (ii) of A follow immediately from (i) and (ii) of B. To show that (iii) of A holds, one can decompose $[0,t]$ equally into n sub–intervals with length t/n. First note that, the probability that there are two or more arrivals in one sub–intervals is $o(t/n)$. Thus, the probability that there are two or more arrivals in any sub–intervals is $o(n\times t/n) = o(1)$. Hence, letting $h = t/n$,

$$\begin{aligned}\mathbb{P}(N(t)=k) &= \mathbb{P}(k \text{ sub–intervals contain exactly one arrival, } n-k \text{ contain none}) + o(1)\\ &= \binom{n}{k}(\lambda h + o(h))^k\,(1-\lambda h + o(h))^{n-k} + o(1)\,. \qquad (1)\end{aligned}$$

Observe that the first part of (1) equals $\mathbb{P}(X_n = k)$ where X_n is a Binomial random variable with parameter $\lambda h + o(h)$. Recall that for any sequence of binomial random variables $\{X_n\}$ satisfying X_n is a binomial random variable with parameter $np_n \to \lambda$ as $n\to\infty$, the distribution of X_n converges to the Poisson distribution with parameter λ. That is, as $n\to\infty$, Equation (1) becomes

$$\mathbb{P}(N(t)=k) = \lim_{n\to\infty} n\lambda h + o(h) = \lambda t\,.$$

By (ii) of B, we have shown (iii) of A.