Statistics 620
Fall, 2009

Name: _____ **UMID #:** _____

Midterm Exam

- **There are** 4 **questions, each worth** 10 **points.**
- **You are allowed a calculator and a single-sided sheet of notes.**
- **Credit will be given for clear explanation and justification, as well as for getting the correct answer.**
- **Cross out any working that you do not wish to be considered as part of your solution. You are advised not to erase unfinished working since partial credit may be available for an indication that an appropriate method was attempted, even if it was later rejected.**

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
Total	40	

1. N guests at a party each put their winter jackets in a big pile. At the end of the party each guest, in turn, picks a randomly chosen jacket from the pile before departing. Let J be the number of guests who pick their own jacket. Find the mean and variance of J. Conjecture what the distribution of J becomes in the limit as $N \to \infty$ (you are not asked to prove this).

Hint: It may help to write I_n for the indicator random variable that the nth guest picks his/her own jacket.

2. Suppose that shocks to a system occur according to a Poisson process with rate λ, and suppose that each shock independently causes the system to fail with probability p. Let N denote the number of shocks that it takes for the system to fail and let T denote the time of failure. Find $\mathbb{P}\big[N=n \,|\, T=t\big]$.

3. Rat and Cat move between two rooms, using different paths. Their motions are independent Markov chains, governed by the transition matrices

$$R = \begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix}.$$

Here, R_{ij} is the chance of Rat being in room j at time $n+1$ given that he is in room i at time n; C_{ij} is the chance of Cat being in room j at time $n+1$ given that she is in room i at time n. Suppose Cat starts in room 1 and Rat starts in room 2. If they are ever in the same room, then Cat catches Rat. How long, on average, will this take?

4. Let $N(t)$ be a renewal process whose inter-arrival times $X_1, X_2, \ldots$ have distribution F and expectation μ. Let $m(t) = \mathbb{E}[N(t)]$. Show that

$$\lim_{t\to\infty} \inf_{s>t} \frac{m(t)}{t} \geq \frac{1}{\mu}.$$

Hint: Recall Wald's equation, that if N is a stopping time for $X_1, X_2, \ldots$ then $\mathbb{E}\big[\sum_{i=1}^{N} X_i\big] = \mathbb{E}[N]\,\mathbb{E}[X]$. You may use this result without proof, or you may attempt to prove it for extra credit if time permits.