Statistics 620
Miterm exam, Fall 2009

1. N guests at a party each put their winter jackets in a big pile. At the end of the party each guest, in turn, picks a randomly chosen jacket from the pile before departing. Let J be the number of guests who pick their own jacket. Find the mean and variance of J. Conjecture what the distribution of J becomes in the limit as $N \to \infty$ (you are not asked to prove this).

Hint: It may help to write I_n for the indicator random variable that the nth guest picks his/her own jacket.

Solution: Define I_n as the indicator random variable that the nth guest picks his/her own jacket, then

$$I_n = \begin{cases} 1 & \text{w.p } \frac{1}{N} \\ 0 & \text{w.p } \frac{N-1}{N} \end{cases}$$

It follows that $J = \sum_{n=1}^{N} I_n$.

$$\begin{aligned}
\mathbb{E}\big[J\big] &= \mathbb{E}\Big[\sum_{n=1}^{N} I_n\Big] = 1 \\
\mathrm{Var}\big[I_n\big] &= \mathbb{E}\big[I_n^2\big] - \mathbb{E}\big[I_n\big]^2 \\
&= \frac{1}{N} - \frac{1}{N^2} = \frac{(N-1)}{N^2} \\
\text{For } i \neq j, \quad \mathrm{Cov}(I_i, I_j) &= \mathbb{E}\big[I_i I_j\big] - \mathbb{E}\big[I_i\big]\mathbb{E}\big[I_j\big] \\
&= \frac{1}{N(N-1)} - \frac{1}{N^2} \\
\mathrm{Var}\big[J\big] &= \mathrm{Var}\big[\sum_{n=1}^{N} I_n\big] \\
&= \sum_{n=1}^{N} \mathrm{Var}\Big[I_n\Big] + 2\sum_{i<j} \mathrm{Cov}(I_i, I_j) \\
&= N\frac{(N-1)}{N^2} + N(N-1)\left(\frac{1}{N(N-1)} - \frac{1}{N^2}\right) \\
&= 1
\end{aligned}$$

Note that $I_1, \ldots, I_N$ are weakly correlated, approaching independence as $N \to \infty$. Thus we conjecture that the distribution of J matches the limit of Binomial $(N, 1/N)$, i.e., J approaches Poisson(1) as $N \to \infty$. In addition, the mean and variance of J are correct for this limit.

2. Suppose that shocks to a system occur according to a Poisson process with rate λ, and suppose that each shock independently causes the system to fail with probability p. Let N denote the number of shocks that it takes for the system to fail and let T denote the time of failure. Find $\mathbb{P}\big[N{=}n \,|\, T{=}t\big]$.

Solution: Define

$$\begin{cases} N(t) &= \text{number of shocks by time } t \\ N_1(t) &= \text{number of shocks that do not cause failure by time } t \\ N_2(t) &= \text{number of shocks that cause failure by time } t \end{cases}$$

It follows from the splitting property of Poisson processes that $\{N_1(t)\}$, $\{N_2(t)\}$ are independent Poisson processes with rate $\lambda(1-p)$ and λp, respectively. So, $N_1(t)$ is independent of $\{T=t\}$ and

$$\begin{aligned} \mathbb{P}\big[N{=}n\,|\,T{=}t\big] &= \mathbb{P}\big[N_1(t)=n-1, dN_2(t)=1\,|\,T{=}t\big] \\ &= \mathbb{P}\big[N_1(t)=n-1\big] \\ &= \frac{\exp^{-\lambda(1-p)t}[\lambda(1-p)t]^{n-1}}{(n-1)!} \end{aligned}$$

3. Rat and Cat move between two rooms, using different paths. Their motions are independent Markov chains, governed by the transition matrices

$$R = \begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix}.$$

Here, R_{ij} is the chance of Rat being in room j at time $n+1$ given that he is in room i at time n; C_{ij} is the chance of Cat being in room j at time $n+1$ given that she is in room i at time n. Suppose Cat starts in room 1 and Rat starts in room 2. If they are ever in the same room, then Cat catches Rat. How long, on average, will this take?

Solution: Notice that, at each time point, the only possibilities are that (i) both animals remain in their current rooms; (ii) Cat and Rat switch rooms; (iii) Cat catches Rat. This leads us to define

$$\begin{aligned} \mu_1 &= \mathbb{E}\big[\text{time to catch } \,|\,\text{cat starts in room 1 and rat starts in room 2}\big] \\ \mu_2 &= \mathbb{E}\big[\text{time to catch } \,|\,\text{cat starts in room 2 and rat starts in room 1}\big] \end{aligned}$$

Then, conditioning on the 1st transition, we have

$$\mu_1 = (C_{11}R_{21} + C_{12}R_{22}) + C_{11}R_{22}(1+\mu_1) + C_{12}R_{21}(1+\mu_2) = 0.38 + 0.06(1+\mu_1) + 0.56(1+\mu_2)$$

$$\mu_2 = (C_{21}R_{11} + C_{22}R_{12}) + c_{21}R_{12}(1+\mu_1) + C_{22}R_{11}(1+\mu_2) = 0.44 + 0.48(1+\mu_1) + 0.08(1+\mu_2)$$

Solving this pair of linear equations, we get $\mu_1 \approx 2.5$.

4. Let $N(t)$ be a renewal process whose inter-arrival times $X_1, X_2, \ldots$ have distribution F and expectation μ. Let $m(t) = \mathbb{E}[N(t)]$. Show that

$$\lim_{t\to\infty} \inf_{s>t} \frac{m(t)}{t} \geq \frac{1}{\mu}.$$

Hint: Recall Wald's equation, that if N is a stopping time for $X_1, X_2, \ldots$ then $\mathbb{E}\big[\sum_{i=1}^{N} X_i\big] = \mathbb{E}[N]\,\mathbb{E}[X]$. You may use this result without proof, or you may attempt to prove it for extra credit if time permits.

Solution: This is in the notes.