Homework 1 (Stat 620, Fall 2009)

Due Tue Sept 15, in class

1. **(a)** Let N denote a nonnegative integer-valued random variable. Show that

$$\mathbb{E}[N] = \sum_{k=1}^{\infty} \mathbb{P}\{N \geq k\} = \sum_{k=0}^{\infty} \mathbb{P}\{N > k\}.$$

(b) In general show that if X is nonnegative with distribution F, then

$$\begin{aligned} \mathbb{E}[X] &= \int_0^\infty \overline{F}(x)\,dx, \\ \mathbb{E}[X^n] &= \int_0^\infty n x^{n-1} \overline{F}(x)\,dx. \end{aligned}$$

Comment: These identities will be useful later in the course.

2. Let X_n denote a binomial random variable, $X_n \sim \text{Binomial}\,(n, p_n)$ for $n \geq 1$. If $np_n \to \lambda$ as $n \to \infty$, show that

$$\mathbb{P}\{X_n = i\} \to e^{-\lambda}\lambda^i/i! \qquad \text{as } n \to \infty.$$

Hint: Write out the required binomial probability, expanding the binomial coefficient into a ratio of products. Taking logarithms may be helpful to show that $\lim_{n\to\infty} c_n = c$ implies $\lim_{n\to\infty}(1 - c_n/n)^n = e^{-c}$.

3. Let F be a continuous distribution function and let U be a uniformly distributed random variable, $U \sim \text{Uniform}\,(0, 1)$.

(a) If $X = F^{-1}(U)$, show that X has distribution function F.

(b) Show that $-\log(U)$ is an exponential random variable with mean 1.

Comment: Part (b) gives a way to simulate exponential random variables using a computer with a random number generator producing $U[0, 1]$ random variables.

4. Let $f(x)$ and $g(x)$ be probability density functions, and suppose that for some constant c, $f(x) \leq cg(x)$ for all x. Suppose we can generate random variables having density function g, and consider the following algorithm.

Step 1. Generate Y, a random variable having density function g.

Step 2. Generate $U \sim \text{Uniform}\,(0, 1)$.

Step 3. If $U \leq \frac{f(Y)}{cg(Y)}$ set $X = Y$. Otherwise, go back to Step 1.

Assuming that successively generated random variables are independent, show that:

(a) X has density function f.

(b) the number of iterations of the algorithm needed to generate X is a geometric random variable with mean c.

Comment: The procedure investigated in this problem is a standard computational tool for simulating a random variable with a given target density $f(x)$.

5. If $X_1, X_2, \ldots, X_n$ are independent and identically distributed exponential random variables with parameter λ, show that $S = \sum_{i=1}^{n} X_n$ has a gamma distribution with parameters (n, λ). That is, show that the density function of S is given by

$$f(t) = \lambda e^{-\lambda t}(\lambda t)^{n-1}/(n-1)!, \qquad t \geq 0.$$

Instruction: Use moment generating functions for this question.

Recommended reading:
Chapter 1 of Ross "Stochastic Processes" is all relevant material. There are too many examples to study them all! Some suggested examples are 1.3(A), 1.3(C), 1.5(A), 1.5(D), 1.9(A).

Supplementary exercises: 1.5, 1.22.
These are optional, but recommended. Do not turn in solutions—they are in the back of the book. To prove Booles's inequality (B5 in the notes for the first class), one can write $\bigcup_{i=1}^{\infty} E_i$ as a disjoint union via defining $F_i = E_i \cap E_{i-1}^c \cap \cdots \cap E_1^c$.