Homework 2 (Stat 620, Fall 2009)

Due Thu Sept 24, in class

[THE DUE DATE IS EXTENDED FROM TUES TO COVER NECESSARY MATERIAL]

1. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Calculate $\mathbb{E}\big[N(t)N(t+s)\big]$.

   Comment: Please state carefully where you make use of basic properties of Poisson processes, such as stationary, independent increments.

2. Suppose that $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes with rates λ_1 and λ_2. Show that $\{N_1(t)+N_2(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1+\lambda_2$. Also, show that the probability that the first event of the combined process comes from $\{N_1(t), t \geq 0\}$ is $\lambda_1/(\lambda_1+\lambda_2)$, independently of the time of the event.

3. Buses arrive at a certain stop according to a Poisson process with rate λ. If you take the bus from that stop then it takes a time R, measured from the time at which you enter the bus, to arrive home. If you walk from the bus stop the it takes a time W to arrive home. Suppose that your policy when arriving at the bus stop is to wait up to a time s, and if a bus has not yet arrived by that time then you walk home.
   (a) Compute the expected time from when you arrive at the bus stop until you reach home.
   (b) Show that if $W < 1/\lambda + R$ then the expected time of part (a) is minimized by letting $s = 0$; if $W > 1/\lambda + R$ then it is minimizes by letting $s = \infty$ (that is, you continue to wait for the bus); and when $W = 1/\lambda + R$ all values of s give the same expected time.
   (c) Give an intuitive explanation of why we need only consider the cases $s = 0$ and $s = \infty$ when minimizing the expected time.

4. Cars pass a certain street location according to a Poisson process with rate λ. A person wanting to cross the street at that location waits until she can see that no cars will come by in the next T time units. Find the expected time that the person waits before starting to cross. (Note, for instance, that if no cars will be passing in the first T time units then the waiting time is 0.)

   Comment: An elegant approach is to condition on the first arrival time.

5. Individuals enter a system in accordance with a Poisson process having rate λ. Each arrival independently makes its way through the states of the system. Let $\alpha_i(s)$ denote the probability that an individual is in state i a time s after it arrived. Let $N_i(t)$ denote the number of individuals in state i at time t. Show that the $N_i(t), i \geq 1$, are independent and $N_i(t)$ is Poisson with mean equal to

   $$\lambda\, \mathbb{E}\big[\text{amount of time an individual is in state } i \text{ during its first } t \text{ units in the system}\big].$$

   Comment:You will probably want to make use of Theorem 2.3.1 of Ross. This question is similar to a multivariate version of Proposition 2.3.2, and you may need the multinomial distribution. If n independent experiments each give rise to outcomes $1, \ldots, r$ with respective probabilities $p_1, \dots, p_r$, and X_i counts the number of outcomes of type i, then $X_1, \dots, X_r$ are *multinomial.* For $\sum_{i=1}^r n_i = n$,

$$\mathbb{P}(X_1 = n_1, \dots, X_r = n_r) = \frac{n!}{n_1! \dots n_r!} \prod_{i=1}^r p_i^{n_i}.$$

Recommended reading:
Sections 2.1 through 2.4, excluding 2.3.1.

Supplementary exercises: 2.14, 2.22.
These are optional, but recommended. Do not turn in solutions—they are in the back of the book.