Homework 3 (Stat 620, Fall 2009)

Due Thu Oct 1, in class

1. Prove the renewal equation

$$m(t) = F(t) + \int_0^t m(t-x)\, dF(x)$$

Hint: One approach is to use the identity $\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|Y]\big]$ for appropriate choices of X and Y.

2. Prove that the renewal function $m(t), 0 \leq t < \infty$ uniquely determines the interarrival distribution F.
Hint: Laplace transforms may be useful.

3. Let $\{N(t), t \geq 0\}$ be a renewal process and suppose that for all n and t, conditional on the event that $N(t) = n$, the event times $S_1, \ldots, S_n$ are distributed as the order statistics of a set of independent uniform $(0, t)$ random variables. Show that $\{N(t), \geq 0\}$ is a Poisson process.
Hint: Consider $\mathbb{E}[N(s) \mid N(t)]$ and then use the result of Problem 2.

4. The random variables $X_1, \ldots, X_n$ are said to be exchangeable if $X_{i_1}, \ldots, X_{i_n}$ has the same joint distribution as $X_1, \ldots, X_n$ whenever $i_1, i_2, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$. That is, they are exchangeable if the joint distribution function $\mathbb{P}\{X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n\}$ is a symmetric function of $(x_1, x_2, \ldots, x_n)$. Let $X_1, X_2, \ldots$ denote the interarrival times of a renewal process.
(a) Argue that conditional on $N(t) = n$, $X_1, \ldots, X_n$ are exchangeable. Would $X_1, \ldots, X_n, X_{n+1}$ be exchangeable (conditional on $N(t) = n$)?
(b) Use (a) to prove that for $n > 0$

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_{N(t)}}{N(t)} \middle| N(t) = n\right] = \mathbb{E}[X_1 | N(t) = n].$$

(c) Prove that

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_{N(t)}}{N(t)} \middle| N(t) > 0\right] = \mathbb{E}[X_1 | X_1 < t].$$

Hint: (a) One approach to showing this formally involves writing $\mathbb{P}[X_1 \leq x_1, \ldots, X_n \leq x_n, N(t) = n]$ as a multiple integral against the joint distribution function $F_{X_1 \ldots X_{n+1}}(y_1, \ldots, y_{n+1})$.

5. Consider a miner trapped in a room that contains three doors. Door 1 leads her to freedom after two-days' travel; door 2 returns her to her room after four-days' journey; and door 3 returns her to her room after eight-days' journey. Suppose at all times she is equally to choose any of the three doors, and let T denote the time it takes the miner to become free.
(a) Define a sequence of independent and identically distributed random variables $X_1, X_2, \ldots$ and a stopping time N such that

$$T = \sum_{i=1}^{N} X_i.$$

Note: You may have to imagine that the miner continues to randomly choose doors even after she reaches safety.

(b) Use Wald's equation to find $\mathbb{E}[T]$.

(c) Compute $\mathbb{E}\big[\sum_{i=1}^{N} X_i | N = n\big]$ and note that it is not equal to $\mathbb{E}\big[\sum_{i=1}^{n} X_i\big]$.

(d) Use part (c) for a second derivation of $\mathbb{E}[T]$.

Recommended reading:
Sections 3.1 through 3.3.

Supplementary exercise: 3.7.
Optional, but recommended. Do not turn in a solution—it is in the back of the book.