# Homework 3 (Stat 620, Fall 2009)

Due Thu Oct 1, in class

1. Prove the renewal equation

$$m(t) = F(t) + \int_0^t m(t-x)\,dF(x)$$

**Hint**: One approach is to use the identity $\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|Y]\big]$ for appropriate choices of $X$ and $Y$.

Solution:

$$\begin{aligned} m(t) &= \mathbb{E}(N(t)) \\ &= \mathbb{E}(\mathbb{E}(N(t)|X_1)) \\ &= \int_0^t \mathbb{E}(N(t)|X_1 = x)\,dF(x) \text{ since } X_1 > t \Rightarrow N(t) = 0 \\ &= \int_0^t \mathbb{E}(1 + N(t-x))\,dF(x) \text{ since renewals are i.i.d.} \\ &= \int_0^t \big[1 + m(t-x)\big]\,dF(x) \\ &= F(t) + \int_0^t m(t-x)\,dF(x)\,. \end{aligned} \tag{1}$$

2. Prove that the renewal function $m(t), 0 \le t < \infty$ uniquely determines the interarrival distribution $F$.
**Hint**: Laplace transforms may be useful.

Solution: Note that there are two definitions of Laplace transform. Under the definition of Ross,

$$\widetilde{F}(s) = \int_0^\infty e^{-st}\,dF(t)\,,$$

and we also have the Laplace transform of the convolution $F * G(t) = \int_0^\infty F(t-s)dG(s)$:

$$\begin{aligned} \widetilde{F * G}(s) &= \int_0^\infty \exp\{-st\} d\left(\int_0^\infty F(t-x)dG(x)\right) = \int_0^\infty \exp\{-st\}\int_0^\infty dF(t-x)dG(x) \\ &= \int_0^\infty \int_x^\infty \exp\{-st\}dF(t-x)dG(x) = \int_0^\infty \int_0^\infty \exp\{-s(t+x)\}dF(t)dG(x) \\ &= \int_0^\infty \exp\{-sx\}\int_0^\infty \exp\{-st\}dF(t)dG(x) \\ &= \int_0^\infty \exp\{-sx\}\widetilde{F}(s)dG(x) = \widetilde{F}(s)\widetilde{G}(s)\,. \end{aligned}$$

Thus the Laplace transform of Equation (1) becomes

$$\widetilde{m}(s) = \widetilde{F}(s) + \widetilde{m}(s)\widetilde{F}(s)$$

so

$$\widetilde{F}(s) = \frac{\widetilde{m}(s)}{1+\widetilde{m}(s)}\,. \tag{2}$$

By the uniqueness of Laplace transforms, $\widetilde{m}(s)$ uniquely determines $F$. Another way to obtain Relation (2) is to calculate the Laplace transform of the identity

$$m(t) = \sum_{n=1}^{\infty} F_n(t)\,,$$

where $F_n$ is the $n$-th convolution of $F$ and $\widetilde{F}_n(s) = \widetilde{F}^n(s)$. If we use another definition of Laplace transform:

$$\widetilde{F}(s) = \int_0^{\infty} e^{-st} F(t) dt\,,$$

the calculation becomes slightly different. In particular, the Laplace transform of $\int_0^t m(t-s)dF(s)$ becomes $\widetilde{m}(s)s\widetilde{F}(s)$. In this case, the Relation (2) becomes

$$\widetilde{F}(s) = \frac{\widetilde{m}(s)}{1+s\widetilde{m}(s)}\,.$$

3. Let $\{N(t), t \geq 0\}$ be a renewal process and suppose that for all $n$ and $t$, conditional on the event that $N(t) = n$, the event times $S_1, \dots, S_n$ are distributed as the order statistics of a set of independent uniform $(0,t)$ random variables. Show that $\{N(t), \geq 0\}$ is a Poisson process.
**Hint**: Consider $\mathbb{E}[N(s) \mid N(t)]$ and then use the result of Problem 2.
<u>Solution</u>: Following the hints

$$\mathbb{E}[N(s)|N(t) = n] = \mathbb{E}[\sum_{i=1}^{n} \mathbf{1}(U_{(i)} \leq s)]$$

where $U_{(1)}, \cdots, U_{(n)}$ are the order statistics of $n$ i.i.d. Unif$[0,t]$ random variables $U_1, \cdots, U_n$. Thus

$$\begin{aligned}
\mathbb{E}[N(s)|N(t) = n] &= \mathbb{E}[\sum_{i=1}^{n} \mathbf{1}(U_{(i)} \leq s)] \\
&= \mathbb{E}[\sum_{i=1}^{n} \mathbf{1}(U_i \leq s)] \text{ since ordering does not affect the sum} \\
&= \sum_{i=1}^{n} \mathbb{P}[U_i \leq s] = ns/t\,.
\end{aligned}$$

Thus

$$m(s) = \mathbb{E}[\mathbb{E}[N(s)|N(t)]] = \frac{s}{t}\mathbb{E}[N(t)] = \frac{s}{t}m(t)\,.$$

The only solution to this is $m(s) = as$ for some constant $a$. This is exactly the renewal function for a rate $a$ Poisson process. Using (3.5) the result follows.

4. The random variables $X_1, \dots, X_n$ are said to be exchangeable if $X_{i_1}, \dots, X_{i_n}$ has the same joint distribution as $X_1, \dots, X_n$ whenever $i_1, i_2, \dots, i_n$ is a permutation of $1, 2, \dots, n$. That is, they are exchangeable if the joint distribution function $\mathbb{P}\{X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n\}$ is a symmetric function of $(x_1, x_2, \dots, x_n)$. Let $X_1, X_2, \dots$ denote the interarrival times of a renewal process.
**(a)** Argue that conditional on $N(t) = n$, $X_1, \dots, X_n$ are exchangeable. Would $X_1, \dots, X_n, X_{n+1}$ be exchangeable (conditional on $N(t) = n$)?
**(b)** Use (a) to prove that for $n > 0$

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_{N(t)}}{N(t)} \big| N(t) = n\right] = \mathbb{E}[X_1 | N(t) = n].$$

**(c)** Prove that

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_{N(t)}}{N(t)} \big| N(t) > 0\right] = \mathbb{E}[X_1 | X_1 < t].$$

**Hint**: (a) One approach to showing this formally involves writing $\mathbb{P}[X_1 \le x_1, \dots, X_n \le x_n, N(t) = n]$ as a multiple integral against the joint distribution function $F_{X_1 \dots X_{n+1}}(y_1, \dots, y_{n+1})$.

<u>Solution</u>: **(a)** Note that

$$\begin{aligned}
\mathbb{P}(X_1 \le x_1, \cdots, X_n \le x_n | N(t) = n) &= \frac{\mathbb{P}(X_1 \le x_1, \cdots, X_n \le x_n, S_n \le t, S_{n+1} > t)}{\mathbb{P}(S_n \le t, S_{n+1} > t)} \\
&= \frac{\mathbb{P}(S_{n+1} > t | X_1 \le x_1, \dots, X_n \le x_n, S_n \le t)\mathbb{P}(X_1 \le x_1, \cdots, X_n \le x_n, S_n \le t)}{\mathbb{P}(S_n \le t, S_{n+1} > t)} \\
&= \frac{\mathbb{P}(S_{n+1} > t | S_n \le t)\mathbb{P}(X_1 \le x_1, \cdots, X_n \le x_n, S_n \le t)}{\mathbb{P}(S_n \le t, S_{n+1} > t)}.
\end{aligned}$$

Now it suffices to show that $\mathbb{P}(X_1 \le x_1, \cdots, X_n \le x_n, S_n \le t)$ is symmetric on $X_1, \dots, X_n$. Indeed,

$$\mathbb{P}(X_1 \le x_1, \dots, X_n \le x_n, S_n \le t) = \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} \mathbf{1}_{\{t \ge \sum_{i=1}^n t_i\}} dF(t_n) \cdots dF(t_2) dF(t_1),$$

where the exchangeability follows by Fubini's theorem. This is not true for $\mathbb{P}(X_1 \le x_1, \dots, X_{n+1} \le x_{n+1} | N(t) = n)$ since obviously, given $N(t) = n$, for $1 \le i \le n$, $X_i$ is bounded while $X_{n+1}$ is unbounded. **(b)** First note that

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_{N(t)}}{N(t)} | N(t) = n\right] = \sum_{i=1}^{n} \frac{1}{n} \mathbb{E}[X_i | N(t) = n].$$

By the exchangeability established in part (a), $\mathbb{E}[X_i | N(t) = n] = \mathbb{E}[X_1 | N(t) = n]$, $i = 1, \cdots, n$. So the required result follows.
**(c)**

$$\begin{aligned}
\mathbb{E}\left[\frac{X_1 + \cdots + X_{N(t)}}{N(t)} | N(t) > 0\right] &= \sum_{n=1}^{\infty} \mathbb{E}[\frac{X_1 + \cdots + X_{N(t)}}{N(t)} | N(t) = n] \mathbb{P}[N(t) = n | N(t) > 0] \\
&= \sum_{n=1}^{\infty} \mathbb{E}[X_1 | N(t) = n] \mathbb{P}[N(t) = n | N(t) > 0] \\
&= \mathbb{E}[X_1 | N(t) > 0] = \mathbb{E}[X_1 | X_1 < t].
\end{aligned}$$

5. Consider a miner trapped in a room that contains three doors. Door 1 leads her to freedom after two-days' travel; door 2 returns her to her room after four-days' journey; and door 3 returns her to her room after eight-days' journey. Suppose at all times she is equally to choose any of the three doors, and let $T$ denote the time it takes the miner to become free.
**(a)** Define a sequence of independent and identically distributed random variables $X_1, X_2, \ldots$ and a stopping time $N$ such that

$$T = \sum_{i=1}^{N} X_i.$$

*Note*: You may have to imagine that the miner continues to randomly choose doors even after she reaches safety.
**(b)** Use Wald's equation to find $\mathbb{E}[T]$.
**(c)** Compute $\mathbb{E}\big[\sum_{i=1}^{N} X_i | N = n\big]$ and note that it is not equal to $\mathbb{E}\big[\sum_{i=1}^{n} X_i\big]$.
**(d)** Use part (c) for a second derivation of $\mathbb{E}[T]$.

Solution: **(a)** Define

$$X = \begin{cases} 2 & \text{Door 1 (probability 1/3)} \\ 4 & \text{Door 2 (probability 1/3)} \\ 8 & \text{Door 3 (probability 1/3)} \end{cases}$$

and $N = \min\{n : X_n = 2\}$. Clearly $N$ is a stopping time as the event $N = n$ is determined by the first $n$ observations of $X$.
**(b)** Using Wald's theorem, $\mathbb{E}[T] = \mathbb{E}[N]\mathbb{E}[X]$. Further $\mathbb{E}[N] = 3$ since $N$ follows a geometric distribution with parameter $p = 1/3$. Also $\mathbb{E}[X] = 14/3$. Thus $\mathbb{E}[T] = 14$.
**(c)**

$$\begin{aligned} \mathbb{E}[\sum_{i=1}^{N} X_i | N = n] &= \mathbb{E}[\sum_{i=1}^{N} X_i | X_1 \neq 2, \cdots, X_{n-1} \neq 2, X_n = 2]\,, \\ &= 2 + (n-1)\mathbb{E}[X_i | X_i \neq 2] = 2 + (n-1)6 = 6n - 4 \\ \mathbb{E}[\sum_{i=1}^{n} X_i] &= n\mathbb{E}[X_i] = 14n/3\,. \end{aligned}$$

**(d)**$\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[\sum_{i=1}^{N} X_i | N]] = \mathbb{E}[6N - 4] = 6 \times 3 - 4 = 14.$

**Recommended reading:**
Sections 3.1 through 3.3.

**Supplementary exercise:** 3.7.
Optional, but recommended. Do not turn in a solution—it is in the back of the book.