

## Homework 4 (Stat 620, Fall 2009)

Due Thu Oct 8, in class

1. Let  $A(t)$  and  $Y(t)$  denote respectively the age and excess at  $t$ . Find:

- (a)  $\mathbb{P}\{Y(t) > x | A(t) = s\}$ .
- (b)  $\mathbb{P}\{Y(t) > x | A(t + x/2) = s\}$ .
- (c)  $\mathbb{P}\{Y(t) > x | A(t + x) > s\}$  for a Poisson process.
- (d)  $\mathbb{P}\{Y(t) > x, A(t) > y\}$ .
- (e) If  $\mu < \infty$ , show that, with probability 1,  $A(t)/t \rightarrow 0$  as  $t \rightarrow \infty$ .

Hint: For (d), use a regenerative process argument (E.g. Ross, section 3.7) to find  $\lim_{t \rightarrow \infty} \mathbb{P}(Y(t) > x, A(t) > y)$ . For (e), you may use without proof the following results on convergence with probability 1: **(L1)**  $\lim_{n \rightarrow \infty} S_n/n = \mu$ ; **(L2)**  $\lim_{t \rightarrow \infty} N(t) = \infty$ ; **(L3)**  $\lim_{t \rightarrow \infty} N(t)/t = 1/\mu$ .

Solution: (a)

$$\begin{aligned} \mathbb{P}[Y(t) > x | A(t) = s] &= \mathbb{P}[X_{N(t)+1} > s + x | X_{N(t)+1} > s] \text{ (by renewal property)} \\ &= \mathbb{P}[X_1 > s + x | X_1 > s] = \bar{F}(s+x)/\bar{F}(s). \end{aligned}$$

Here is a more formal solution:

$$\begin{aligned} \mathbb{P}[Y(t) > x | A(t) = s] &= \mathbb{P}[S_{N(t)+1} > t + x | S_{N(t)} = t - s] = \mathbb{P}[X_{N(t)+1} > s + x | S_{N(t)} = t - s] \\ &= \sum_{n=0}^{\infty} \mathbb{P}[X_{n+1} > s + x | S_n = t - s, N(t) = n] \mathbb{P}[N(t) = n | S_{N(t)} = t - s] \\ &= \sum_{n=0}^{\infty} \mathbb{P}[X_{n+1} > s + x | S_n = t - s, X_{n+1} > s] \mathbb{P}[N(t) = n | S_{N(t)} = t - s] \\ &= \sum_{n=0}^{\infty} \mathbb{P}[X_{n+1} > s + x | X_{n+1} > s] \mathbb{P}[N(t) = n | S_{N(t)} = t - s] \text{ (by independence)} \\ &= \mathbb{P}[X_1 > s + x | X_1 > s] = \bar{F}(s+x)/\bar{F}(s) \end{aligned}$$

(b)

$$\begin{aligned} p &\equiv \mathbb{P}[Y(t) > x | A(t + x/2) = s] \\ &= \mathbb{P}[\text{no events in } [t, t + x] | \text{event at } t + x/2 - s, \text{ no events in } (t + x/2 - s, t + x/2)] \end{aligned}$$

For  $s \geq x/2$ , an argument similar to (a) allows us to write

$$\begin{aligned} p &= \mathbb{P}[\text{no event in } (t + x/2 - s, t + x) | \text{event at } t + x/2 - s, \text{ no events in } (t + x/2 - s, t + x/2)] \\ &= \mathbb{P}[X_{N(t)+1} > s + x/2 | X_{N(t)+1} > s] \mathbb{P}[X_1 > s + x/2 | X_1 > s] = \bar{F}(s+x/2)/\bar{F}(s) \end{aligned}$$

For  $s < x/2$ ,  $\{A(t + x/2) = s\} \Rightarrow \{Y(t) \leq s - x/2\}$ . It follows that  $p = 0$ .

(c)

$$\begin{aligned} q &\equiv \mathbb{P}[Y(t) > x | A(t + x) > s] \\ &= \mathbb{P}[\text{no event in } [t, t + x] | \text{no events in } [t + x - s, t + x]] \end{aligned}$$

for  $0 \leq s \leq x$ , since the process is Poisson with independent increments,

$$q = \mathbb{P}[\text{no event in } [t, t + x - s]] = \exp^{-\lambda(x-s)},$$

where  $\lambda$  is the rate of the Poisson process. For  $s > x$ ,  $q = 1$ .

(d)

$$\begin{aligned} \mathbb{P}(Y(t) > x, A(t) > y) &= \mathbb{P}(X_{N(t)+1} > t - S_{N(t)} + x, t - S_{N(t)} > y) \\ &= \mathbb{P}(X_1 > t + x, t > y | S_{N(t)} = 0) \mathbb{P}(S_{N(t)} = 0) \\ &\quad + \int_0^t \mathbb{P}(X_{N(t)+1} > t - S_{N(t)} + x, t - S_{N(t)} > y | S_{N(t)} = s, X_{N(t)+1} > t - S_{N(t)}) dF_{S_{N(t)}}(s) \\ &= \mathbf{1}_{\{t > y\}} \mathbb{P}(X_1 > t + x | X_1 > t) \mathbb{P}(S_{N(t)} = 0) \\ &\quad + \int_0^t \mathbf{1}_{\{t-s > y\}} \mathbb{P}(X > t + x - s | X > t - s) dF_{S_{N(t)}}(s) \\ &= \mathbf{1}_{\{t > y\}} \bar{F}(t + x) + \int_0^{t-y} \bar{F}(t + x - s) dm(s). \end{aligned}$$

Let  $P_t = \mathbb{P}[Y(t) > x, A(t) > y]$ . Define a regenerative process to be “on” at  $t$  if  $S_{N(t)} < t - y$  and  $S_{N(t)+1} > t + x$ . Thus,  $P_t$  is the probability that the process is “on” at time  $t$ . By the regenerative process limit theorem

$$\begin{aligned} \lim_{t \rightarrow \infty} P_t &= \frac{\mathbb{E}[\text{time “on” during a cycle}]}{\mathbb{E}[\text{time of the cycle}]} \\ &= \frac{\mathbb{E}[\max(X_1 - (x + y), 0)]}{\mu} \\ &= \frac{1}{\mu} \int_{x+y}^{\infty} (z - x - y) dF(z) \end{aligned}$$

(e)

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{A(t)}{t} &= \lim_{t \rightarrow \infty} \frac{t - S_{N(t)}}{t} \\ &= 1 - \lim_{t \rightarrow \infty} \frac{S_{N(t)}}{N(t)} \lim_{t \rightarrow \infty} \frac{N(t)}{t} \\ &= 1 - \mu/\mu \quad (\text{by (L1), (L2) and (L3)}) \\ &= 0 \end{aligned}$$

2. Consider a single-server bank in which potential customers arrive in accordance with a renewal process having interarrival distribution  $F$ . However, an arrival only enters the bank if the server is free when he or she arrives. Would the number of events by time  $t$  constitute a (possibly delayed) renewal process if an event corresponds to a customer:

(a) entering the bank?

(b) leaving the bank?

What if  $F$  were exponential?

Solution: Let  $X_i$  denote the length of the  $i$ -th service and let  $Y_i$  denote the time from the end of  $i$ -th service until the start of the  $i + 1$ -th service. Let  $Y_0$  denote the time when first

arrival enters the bank (and gets service). Note that  $X_i$  and  $Y_i$  may be dependent when the arrival is not a Poisson process.

**(a)** In this case, each cycle consists of  $Z_i = X_i + Y_i, i = 1, 2, \dots$  and  $Z_0 = Y_0$ . Since  $X_i$  and  $Y_i$  are independent of  $X_j$  and  $Y_j$  with  $j = 1, \dots, i - 1$ ,  $\{Z_i\}_{i \in \mathbb{N}}$  are i.i.d. copies. We thus have a delayed renewal process.

**(b)** In this case,  $Z_i = Y_{i-1} + X_i$ . When  $X_i$  and  $Y_i$  are dependent,  $\{Z_i\}_{i \in \mathbb{N}}$  are not i.i.d. copies. We do not have a (delayed) renewal process. One counter example can be constructed as in the sequel. Suppose the service distribution is given by

$$Y_1 = \begin{cases} 1 & \text{w.p. } 0.5 \\ 10 & \text{w.p. } 0.5 \end{cases}$$

and the interarrival times of the customers to the bank  $Z_n \sim F$  are given by,  $Z_1 = 6$  w.p. 1. Then, given a previous interval between departures  $S_n - S_{n-1} = 3$ , we know that the next arrival will enter the bank at time  $S_n + 4$ .

If  $F$  is exponential **(a)** still gives a delayed renewal process. **(b)** now results in a (non-delayed) renewal process, since the memoryless property implies that  $Y_i$  is independent of  $X_i, i \in \mathbb{N}$ . Hence,  $\{Z_i\}_{i \in \mathbb{N}}$  are i.i.d. copies.

3. On each bet a gambler, independently of the past, either wins or loses 1 unit with respective probability  $p$  and  $1 - p$ . Suppose the gambler's strategy is to quit playing the first time she wins  $k$  consecutive bets. At the moment she quits

**(a)** find her expected winnings.

**(b)** find the expected number of bets that she has won.

Hint: It may help you to look at Example 3.5(A) in Ross.

Solution: Let

$$Y_n = \begin{cases} 1 & \text{if } n\text{th game is a win} \\ 0 & \text{else} \end{cases}$$

and

$$X_n = \begin{cases} 1 & \text{if } n\text{th game is a win} \\ -1 & \text{else} \end{cases}$$

and let  $N = \inf\{n \geq k : \sum_{m=n-k+1}^n X_m = k\} = \inf\{n \geq k : \sum_{m=n-k+1}^n Y_m = k\}$ , the first time  $k$  consecutive games are won. Let  $W = \sum_{i=1}^N X_i$ , the gamblers total winnings. Also let  $N_W = \sum_{i=1}^N Y_i$ , the number of games won.

**(a)** From Ross, Example 3.5A,  $\mathbb{E}[N] = \sum_{i=1}^k (1/p)^i$ . Also  $N$  is a stopping time w.r.t  $X_i, i = 1, 2, \dots$ . By Wald's equation, we have

$$\mathbb{E}[W] = \mathbb{E}[N] \mathbb{E}[X_1] = \left( \sum_{i=1}^k \frac{1}{p^i} \right) (2p - 1)$$

**(b)**  $N$  is also a stopping time w.r.t.  $Y_i, i = 1, \dots$ , so Wald's equation gives

$$\mathbb{E}[N_W] = \mathbb{E}[N] \mathbb{E}[Y_1] = \left( \sum_{i=1}^k \frac{1}{p^i} \right) p = \sum_{i=0}^{k-1} \frac{1}{p^i}$$

4. Prove Blackwell's theorem for renewal reward processes. That is, assuming that the cycle distribution is not lattice, show that, as  $t \rightarrow \infty$ ,

$$\mathbb{E}[\text{reward in}(t, t+a)] \rightarrow a \frac{\mathbb{E}[\text{reward incycle}]}{\mathbb{E}[\text{time of cycle}]}.$$

Assume that any relevant function is directly Riemann integrable.

Hint: You may adopt an informal approach by assuming that one can write

$$\mathbb{E}\left[\int_t^{t+a} dR(s)\right] = \int_t^{t+a} \mathbb{E}[dR(s)],$$

and then developing the identity

$$\mathbb{E}[dR(t)] = \mathbb{E}[R_1|X_1 = t]dF(t) + \int_0^t \{\mathbb{E}[R_1|X_1 = t-x]dF(t-x)\} dm(x).$$

If you can find a more elegant or more rigorous solution, that would also be good!

Solution: Let  $R(t)$  be the reward accumulated by time  $t$ . Then,

$$\begin{aligned} \mathbb{E}[\text{reward in}(t, t+a)] &= \mathbb{E}[R(t+a) - R(t)] \\ &= \mathbb{E}\left[\int_t^{t+a} dR(s)\right] \\ &= \int_t^{t+a} \mathbb{E}[dR(s)] \end{aligned}$$

assuming that the interchange is allowed, e.g. if  $R(t)$  is increasing. Now,

$$\begin{aligned} \mathbb{E}[dR(t)] &= \mathbb{E}[\mathbb{E}[dR(t)|S_{N(t)}]] \\ &= \mathbb{E}[dR(t)|S_{N(t)} = 0]\mathbb{P}[S_{N(t)} = 0] + \int_0^\infty \mathbb{E}[dR(t)|S_{N(t)} = y] dF_{S_{N(t)}}(y) \\ &= \mathbb{E}[dR(t)|S_{N(t)} = 0]\bar{F}(t) + \int_0^\infty \mathbb{E}[dR(t)|S_{N(t)} = y]\bar{F}(t-y) dm(y). \end{aligned}$$

Now, since  $R(t)$  only increases when an event occurs,

$$\begin{aligned} \mathbb{E}[dR(t)|S_{N(t)} = y] &= \mathbb{E}[R_{N(t)+1}|X_{N(t)+1} = t-y]dF_{X_{N(t)+1}|S_{N(t)=y}(t-y) \\ &= \mathbb{E}[R_1|X_1 = t-y] \frac{dF(t-y)}{\bar{F}(t-y)}. \end{aligned}$$

This established the identity

$$\mathbb{E}[dR(t)] = \mathbb{E}[R_1|X_1 = t]dF(t) + \int_0^t (\mathbb{E}[R_1|X_1 = t-y]dF(t-y)) dm(x).$$

Now the key renewal theorem gives

$$\lim_{t \rightarrow \infty} \mathbb{E}[dR(t)] = \int_0^t \mathbb{E}[R_1|X_1 = t]dF(t) dt = \mathbb{E}[R_1] dt.$$

Thus

$$\lim_{t \rightarrow \infty} \int_t^{t+a} \mathbb{E}[dR(t)] = \int_t^{t+a} \mathbb{E}[R_1] ds = a \frac{\mathbb{E}[R_1]}{\mu}.$$

**Another approach:** Note that

$$\begin{aligned} \mathbb{E}[\text{reward in } (t, t+a)] &= \mathbb{E}\left[\sum_{n=1}^{N(t+a)} R_n - \sum_{n=1}^{N(t)} R_n\right] \\ &= \mathbb{E}\left[\sum_{n=1}^{N(t+a)+1} R_n\right] - \mathbb{E}\left[\sum_{n=1}^{N(t)+1} R_n\right] + \mathbb{E}[R_{N(t)+1}] - \mathbb{E}[R_{N(t+a)+1}] \end{aligned}$$

Now  $N(t+a)+1$  and  $N(t)+1$  are stopping times for the sequence  $(X_i, R_i), i = 1, \dots$ . Thus from (generalized) Wald's equation

$$\mathbb{E}\left[\sum_{n=1}^{N(t)+1} R_n\right] = \mathbb{E}[N(t)+1]\mathbb{E}[R]$$

and

$$\mathbb{E}\left[\sum_{n=1}^{N(t+a)+1} R_n\right] = \mathbb{E}[N(t+a)+1]\mathbb{E}[R],$$

where  $\mathbb{E}[R]$  is the expected reward in a cycle. Thus

$$\begin{aligned} \mathbb{E}[\text{reward in } (t, t+a)] &= \mathbb{E}[N(t+a)+1]\mathbb{E}[R] - \mathbb{E}[N(t)+1]\mathbb{E}[R] + \mathbb{E}[R_{N(t)+1}] - \mathbb{E}[R_{N(t+a)+1}] \\ &= (m(t+a) - m(t))\mathbb{E}[R] + \mathbb{E}[R_{N(t)+1}] - \mathbb{E}[R_{N(t+a)+1}]. \end{aligned}$$

Now

$$\lim_{t \rightarrow \infty} (m(t+a) - m(t))\mathbb{E}[R] = a\mathbb{E}[R]/\mathbb{E}[X].$$

from Blackwell's theorem. Now, it suffices to show that  $\lim_{t \rightarrow \infty} \mathbb{E}[R_{N(t)+1}]$  exists and is finite. Indeed,

$$\begin{aligned} \mathbb{E}[R_{N(t)+1}] &= \mathbb{E}[R_{N(t)+1} | S_{N(t)} = 0] \bar{F}(t) + \int_0^t \mathbb{E}[R_{N(t)+1} | S_{N(t)} = s] \bar{F}(t-s) dm(s) \\ &= \mathbb{E}[R | X > t] \bar{F}(t) + \int_0^t \mathbb{E}[R | X > t-s] \bar{F}(t-s) dm(s) \\ &= h(t) + \int_0^t h(t-s) dm(s), \end{aligned}$$

where  $h(t) = \mathbb{E}[R | X > t] \bar{F}(t)$ . Then by the Key Renewal theorem, we have

$$\lim_{t \rightarrow \infty} \mathbb{E}[R_{N(t)+1}] = \frac{1}{\mathbb{E}[X]} \int_0^\infty h(s) ds.$$

Here we assumed that  $h(t)$  is directly Riemann integrable.

5. The life of a car is a random variable with distribution  $F$ . An individual has a policy of trading in his car either when it fails or reaches the age of  $A$ . Let  $R(A)$  denote the resale value of an  $A$ -year-old car. There is no resale value of a failed car. Let  $C_1$  denote the cost of a new car and suppose that an additional cost  $C_2$  is incurred whenever the car fails.

(a) Say that a cycle begins each time a new car is purchased. Compute the long-run average cost per unit time.

(b) Say that a cycle begins each time a car in use fails. Compute the long-run average cost per unit time.

*Note:* In both (a) and (b) you are expected to compute the ratio of the expected cost incurred in a cycle to the expected time of a cycle. The answer should, of course, be the same in both parts.

Solution: (a) Clearly,

$$\mathbb{E}[\text{cost per cycle}] = C_1 - \bar{F}(A)R(A) + F(A)C_2$$

and

$$\mathbb{E}[\text{time of cycle}] = \int_0^A x dF(x) + A(1 - F(A)).$$

So, treating the cost as the reward, the renewal reward theorem gives

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[\text{accumulated cost by } t]}{t} = \frac{\mathbb{E}[\text{cost per cycle}]}{\mathbb{E}[\text{time of cycle}]} = \frac{C_1 - \bar{F}(A)R(A) + F(A)C_2}{\int_0^A x dF(x) + A\bar{F}(A)}$$

(b) The chance that a car fails is  $F(A)$ , so the number,  $N$ , of cars bought between failures has the geometric distribution with parameter  $p = F(A)$ . We have,

$$\mathbb{E}[\text{cost per cycle}] = \mathbb{E}[NC_1 - (N - 1)R(A) + C_2] = C_1/F(A) + (1 - 1/F(A))R(A) + C_2$$

and

$$\mathbb{E}[\text{time of cycle}] = \mathbb{E}[(N - 1)A] + \mathbb{E}[\text{car life} | \text{car life} < A] = \bar{F}(A)A/F(A) + \int_0^A x dF(x)/F(A).$$

Thus,

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[\text{accumulated cost by } t]}{t} = \frac{C_1/F(A) + (1 - 1/F(A))R(A) + C_2}{\bar{F}(A)A/F(A) + \int_0^A x dF(x)/F(A)}.$$

Multiplying numerator and denominator by  $F(A)$  gives the same expression as in (a).